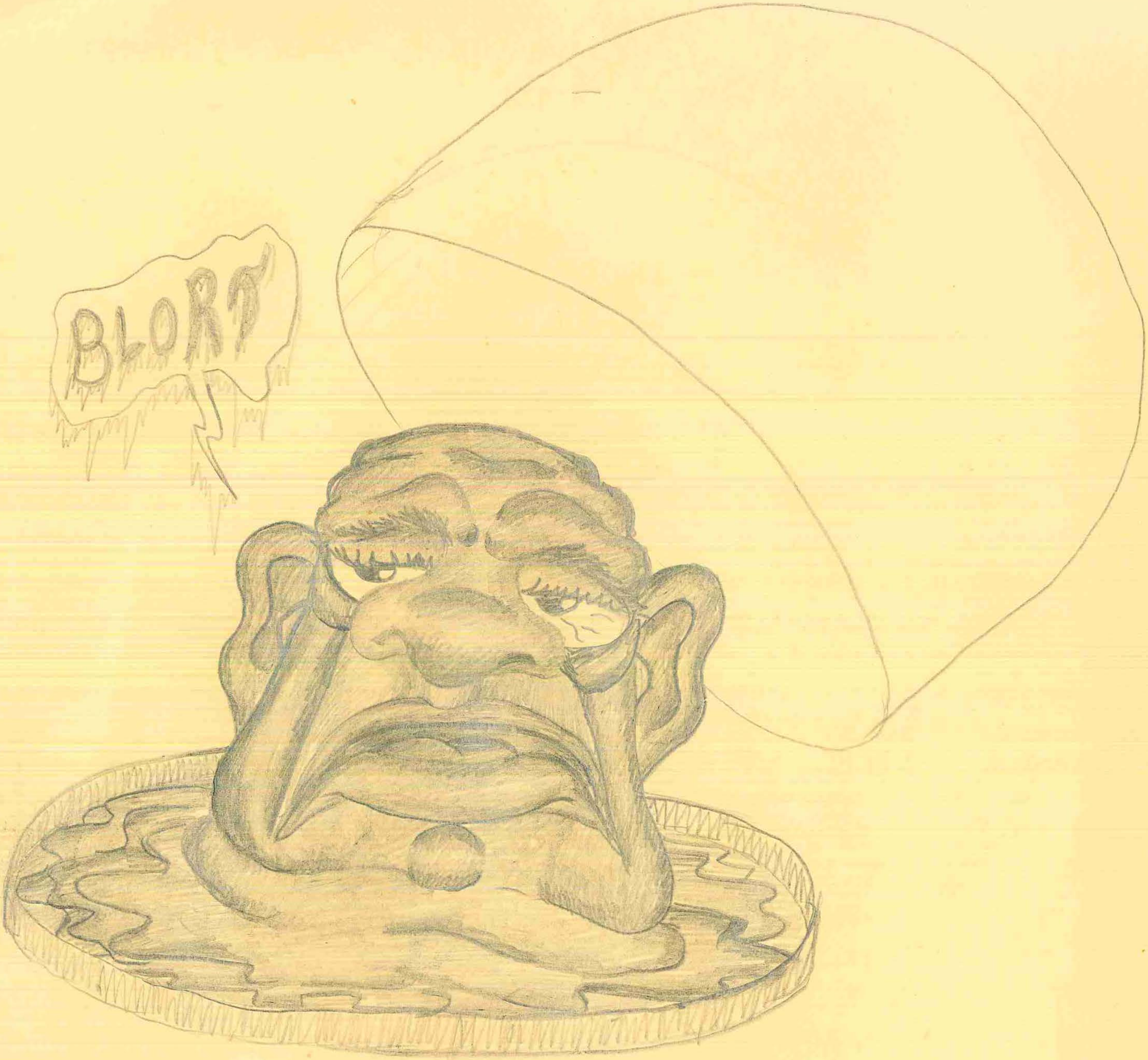
1978

H. OEWISD
ISOPLANICITY +
MEASURE OF
SPATIAL VARIANCE


BLORT

Prof. Dr. A. Lohmann

Physikalisches Institut · Erwin-Rommel-Str. 1 · D 8520 Erlangen

Prof. Dr. R. Marks
University of Washington
Dept. of Electrical Engineering
Seattle, Washington 98195

USA

D 8520 Erlangen, Nov.2, 1978
Erwin-Rommel-Str. 1
Telefon 09131/85

Unser Zeichen

Lo/rp

Dear Bob,

Your long letter and the reprints did arrive. Thanks. My detailed answer will take some time due to the beginning semester.

Thanks again and best regards,

Adolf

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

18 September 1978

Dr. John F. Walkup
Department of Electrical Engineering
Texas Tech University
Lubbock, Texas 79409

John,

Enclosed are copies of three unpublished blurbs on the measure of spatial invariance:

1. "On the Convergence of the PIA" presents some interesting ideas in an elementary Hilbert (signal) space context.
2. The second paper presents a neat Cantorian view of the PIA.
3. The third paper is on the Lohmann-Paris invariance measure.

There were two ideas I had on measuring spatial variance:

1. Expansion of the line-spread function, $h(x-\xi,\xi)$, about some point $\xi = \hat{\xi}$ in a Taylor series:

$$h(x;\xi) = \sum_{n=o}^{\infty} \frac{(\xi-\hat{\xi})^n}{n!} \left(\frac{\delta}{\delta\xi}\right)^n h(x;\hat{\xi}) = h(x;\hat{\xi}) + R(x,\xi,\hat{\xi}) \ ,$$

where the variance "residue" is

$$R(x,\xi,\hat{\xi}) = \sum_{n=1}^{\infty} \frac{(\xi-\hat{\xi})^n}{n!} \left(\frac{\delta}{\delta\xi}\right)^n h(x;\hat{\xi}) \ .$$

For the invariant case, the residue is identically zero for all sample points $\hat{\xi}$. It would seem that some operation on R could lead to a spatial variance measure. We are, of course, limited to line-spread functions which in some sense are analytical in ξ for a given x.

2. The second idea is a generalization of the variation bandwidth concept. A bandwidth is a measure of the dispersion of the line-spread function with respect to its variation variable. In prabability, a pdf's "range" is analogous in concept to a spectrum's bandwidth. A second popular measure of dispersion in the field of proabability is "variance". Using the variation spectrum, define the "variance" in ν as

$$\sigma(x) = \int_{-\infty}^{\infty} |\nu H_{\xi}(x;\nu)|^2 \, d\nu \ .$$

Using Parceval's theorem:

$$\sigma(x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} |\frac{d}{d\xi} h(x,\xi)|^2 \, d\xi \ .$$

Obviously, $\sigma(x) = 0$ for the space-invariant case.

There are a number of possible extensions and generalizations of this concept. We could, for example, define

$$\sigma_n(x) = \frac{1}{(2\pi)^2} \int_{\ell_n}^{\ell_{n+1}} |\frac{d}{d\xi} h(x;\xi)|^2 \, d\xi \ ,$$

where ℓ_n and ℓ_{n+1} define an isoplanatic patch's endpoints. The quantity

$$\theta_n = \int_{-\infty}^{\infty} \sigma_n(x) \, dx$$

can then be intepreted as the variance measure of the patch. We could calibrate the input plane by choosing the ℓ_n's such that θ_n has the same value for all n.

There are a number of further possibilities. We can, for example, formulate a measure of the contribution of the n^{th} input patch to the output interval $k_m \leq x \leq k_{m+1}$. This would be

$$\theta_{nm} = \int_{k_m}^{k_{m+1}} \sigma_n(x) \, dx \ .$$

Note that

$$\theta_n = \sum_m \theta_{nm} .$$

There are number of possible alternatives. I have explored none in depth. I do, however, have an idea that good examples will be obtained primarily with physical finite energy line-spread functions.

Hope this will be of help to you.

Best regards,

Bob

Robert J. Marks II
Assistant Professor

RM:bb
enclosures

ON THE CONVERGENCE OF THE PIECEWISE ISOPLANATIC APPROXIMATION

The authors have presented a model by which linear space-variant system outputs may be approximated by dividing the input plane into a number of isoplanatic patches(1). If a linear system has a response of $h(x-\xi;\xi)$ to an input $\delta(x-\xi)$ where $\delta(x)$ is the Dirac delta, then the system output, $g_o(x)$, due to an input $g_i(\xi)$ is given through the superposition integral as

$$g_o(x) = \int_{-\infty}^{\infty} g_i(\xi)\, h(x-\xi;\xi)\, d\xi \tag{1}$$

If the system input plane is divided into m isoplanatic patches, the n^{th} of which extends from l_n to l_{n+1}, then the piecewise isoplanatic approximation to the true output is

$$\tilde{g}_o(x) = \sum_{n=1}^{m} \int_{l_n}^{l_{n+1}} g_i(\xi)\, h(x-\xi;x_n)\, d\xi \tag{2}$$

where

$$l_n \leq x_n \leq l_{n+1} \tag{3}$$

I. On the Absolute Convergence of the Piecewise Isolpanatic Approximation

In the development of the piecewise isoplanatic approximation (PIA), the authors made the erroneous statement that

$$\lim_{m\to\infty} \tilde{g}_o(x) = g_o(x) \tag{4}$$

Although true in a sampled sense, Eq 4 is not exactly true.

The invalidity of Eq 4 may be shown by first writting

$$\lim_{m\to\infty} \tilde{g}_o(x) = \lim_{m\to\infty} \sum_{n=1}^{m} \int_{\ell_n}^{\ell_{n+1}} g_i(\xi)\, h(x-\xi; x_n)\, d\xi$$

$$= \int_{-\infty}^{\infty} g_i(\xi) \lim_{m\to\infty} \sum_{n=1}^{m} h(x-\xi; x_n)\, \mu(\xi-\ell_n) \times \mu(-\xi+\ell_{n+1})\, d\xi \tag{5}$$

where $\mu(x)$, the unit step function, is defined as

$$\mu(x) = \begin{cases} 1 \; ; & x \geq 0 \\ 0 \; ; & x < 0 \end{cases} \tag{6}$$

Comparing Eqs 5 and 1, we see that Eq 4 is true only if

$$h(x-\xi;\xi) = \lim_{m\to\infty} \sum_{n=1}^{m} h(x-\xi;\chi_n)\mu(\xi-l_n)\mu(l_{n+1}-\xi) \tag{7}$$

We will now show that Eq 7 is in fact not a valid statement.

Consider Fig. 1 in which a function $f(\xi)$, zero outside the interval

$$a = l_1 \leq \xi \leq b = l_{m+1} \tag{8}$$

is represented in a sampled manner as

$$\sum_{n=1}^{m} f(\chi_n)\mu(\xi-l_n)\mu(l_{n+1}-\xi) \tag{9}$$

The patch points l_n and χ_n are as previously defined. In order to disprove Eq 7, we must now prove

$$f(\xi) \neq \lim_{m\to\infty} \sum_{n=1}^{m} f(\chi_n)\mu(\xi-l_n)\mu(l_{n+1}-\xi) \tag{10}$$

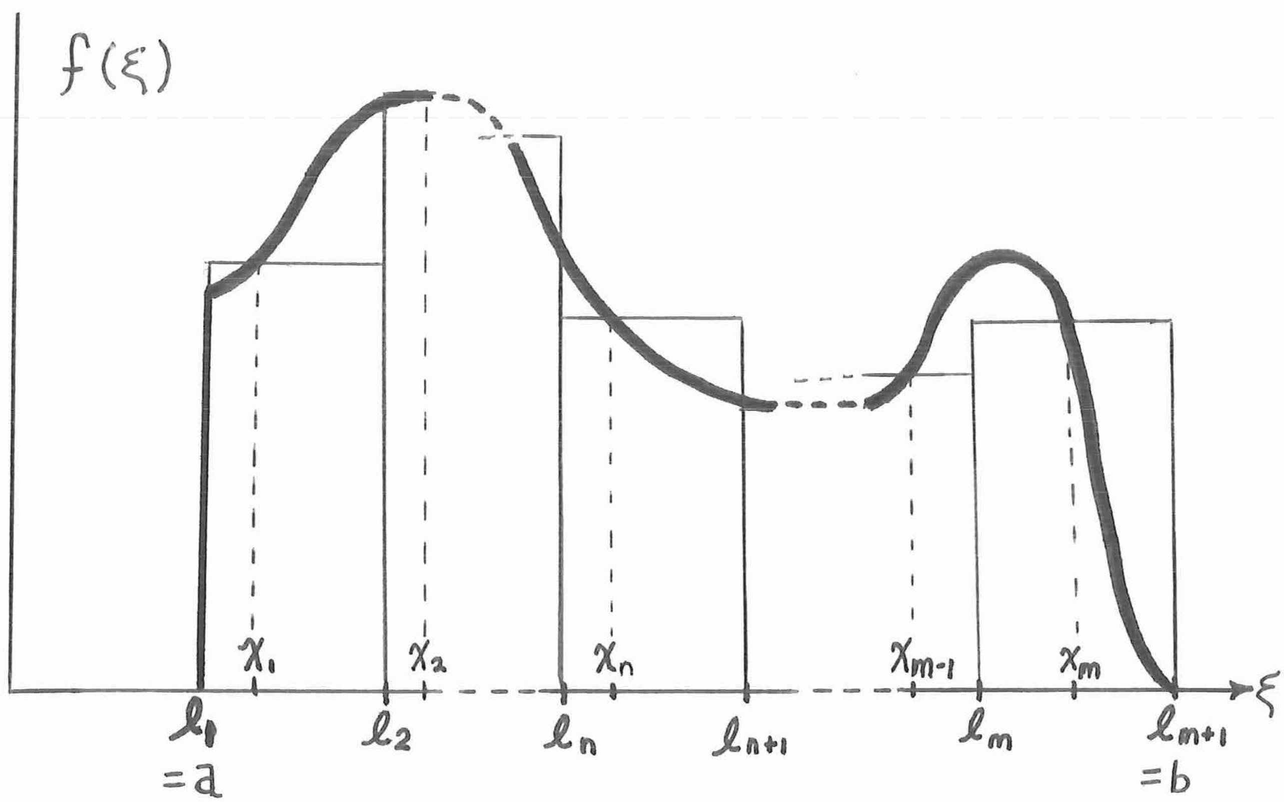


Fig 1: Division of the function $f(\xi)$ into m pulses weighted by the function. As m becomes arbitrarily large, the pulse representation does not approach $f(\xi)$.

To do this, only one counter example needs to be shown. As such, let

$$f(\xi) = \xi\, \mu(\xi+1)\, \mu(1-\xi) \tag{11}$$

As pictured in Fig. 2, $f(\xi)$ is a straight line with unity slope on the interval (-1,1). Since no patch division is cited, we are free to choose our own. If all the intervals are chosen to have width Δ, then

$$m\Delta = 2 \tag{12}$$

Thus

$$\ell_n = n\Delta = \frac{2n}{m} \tag{13}$$

We also arbitrarily let

$$x_n = (n + \tfrac{1}{2})\Delta \tag{14}$$

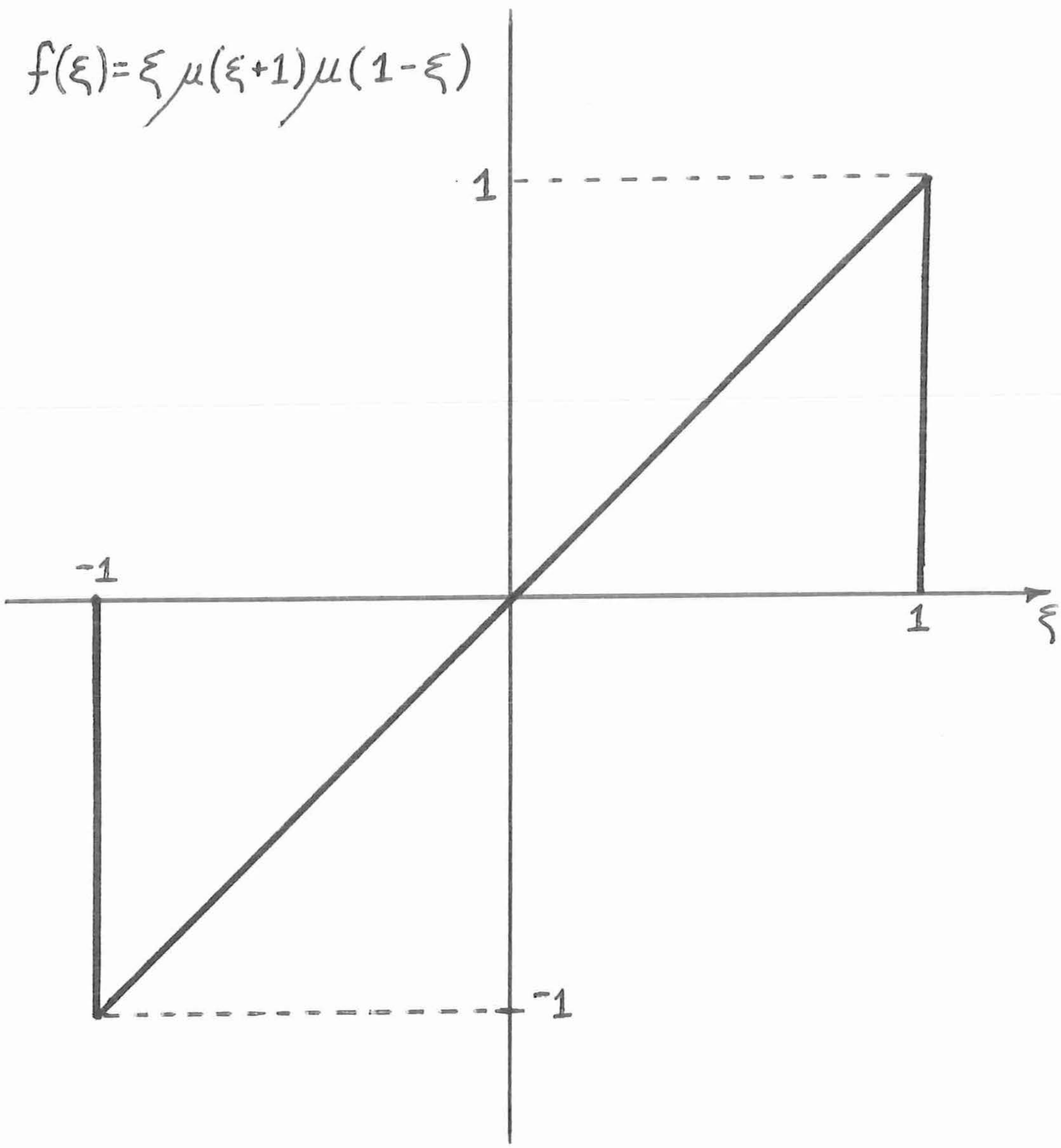


Fig 2: Relationship employed to disprove Eq 7.

Therefore

$$\sum_{n=1}^{m} f(x_n)\mu(\xi-\ell_n)\mu(\ell_{n+1}-\xi)$$

$$= \sum_{n=1}^{m} \frac{2n+1}{m}\mu\left(\frac{2n+1}{m}-1\right)\mu\left(1-\frac{2n+1}{m}\right) \times \mu\left(\xi-\frac{2n}{m}\right)\mu\left(\frac{2(n+1)}{m}-\xi\right) \tag{15}$$

Each term in the above sum is either $(2n+1)/m$ or 0. Since both n and m are integers, $(2n+1)/m$ is a rational number. In that sums of rational numbers are rational, Eq 15, even in the limit, can take on only rational values. Observing that $f(\xi)$ takes on all rational and irrational values between -1 and 1 proves our claim that Eq 10 is generally correct and that the convergence claims made by Eqs 7 and 4 are in fact incorrect.

The crux of the lack of absolute convergence of $\tilde{g}_o(x)$ to $g_o(x)$ as m goes to infinity lies in the order of infinity of system input-output relationships. For the true case, each point on the input plane is essentially assigned a unique output thereby constituting an unaccoutably infinite of defining relationships. The PIA has only a countably infinite number of such relationships in the limit. Some consequences of this non-convergence are now illustrated via example.

II. Representation of the True and PIA Outputs in Signal Space

The true and PIA outputs may be represented as points on orthonormal axes in signal space (2). As such, let

$$\phi(x) = \frac{1}{\sqrt{E}}\, g_o(x) \tag{16}$$

where

$$E = \int_{-\infty}^{\infty} |g_o(x)|^2 dx \tag{17}$$

Also, let

$$\tilde{E} = \int_{-\infty}^{\infty} |\tilde{g}_o(x)|^2 dx \tag{18}$$

If

$$\tilde{g}_o(x) = \alpha\, \phi(x) + \tilde{\alpha}\, \tilde{\phi}(x) \tag{19}$$

where $\phi(x)$ and $\tilde{\phi}(x)$ are orthonormal, then

$$\begin{aligned} \alpha &= \int_{-\infty}^{\infty} \tilde{g}_o(x)\, \phi^*(x)\, dx \\ &= \frac{1}{\sqrt{E}} \int_{-\infty}^{\infty} \tilde{g}_o(x)\, g_o^*(x)\, dx \end{aligned} \tag{20}$$

With knowledge of E, $\tilde{E}$, and α, we may view the relationship of $\tilde{g}_o(x)$ to $g_o(x)$ as pictured in Fig. 3. Note that, in most instances, E and α, as well as $\tilde{\phi}(x)$ are functions of the input plane calibration parameters l_n and χ_n.

A. Illustration of Convergence

As an example of signal space illustration of the non-convergence of the PIA, consider the ideal magnifier with an input-output relationship of

$$g_o(x) = \frac{1}{M} g_i\left(\frac{x}{M}\right) \tag{21}$$

From Eq 17

$$E = \frac{1}{M} \int_{-\infty}^{\infty} |g_i(\xi)|^2 d\xi \tag{22}$$

The PIA of the ideal magnifier (1) can be shown to be

$$\tilde{g}_o(x) = \sum_{n=1}^{m} g_i[x-(M-1)\chi_n] \, \mu[x-l_n-(M-1)\chi_n] \times \mu[-x+l_{n+1}+(M-1)\chi_n] \tag{23}$$

If attention is restricted to the case where $M>1$, then $\tilde{g}_o(x)$ is recognized as a non-overlapping piecewise shifted version of $g_i(x)$. Thus

$$\tilde{E} = \int_{-\infty}^{\infty} |g_i(\xi)|^2 d\xi = ME \tag{24}$$

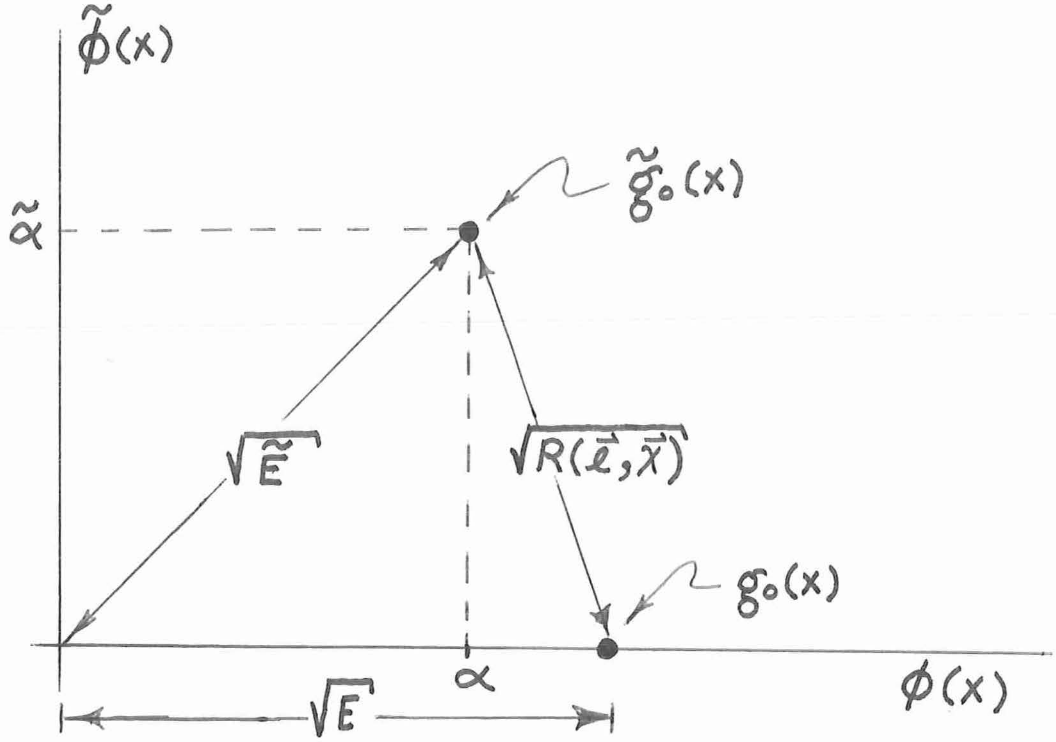


Fig 3: The true and PIA outputs as represented in signal space.

As pictured if Fig. 4, the locus of all possible PIA representations of the magnifier for $M > 1$ forms a circle in signal space centered at the origin and with radius ME. The non-convergence of $\tilde{g}_o(x)$ to $g_o(x)$ is illustrated by the non-intersection of this locus to the true output point.

A physical example of the non-convergence of the magnifier's PIA with the corresponding input and true output is offered in Fig. 5. One sees that as the patch density becomes arbitrarily large, the PIA does not approach the true output in the strict sense although attempts at mimicking $g_o(x)$ are obvious.

B. Input Plane Calibration Optimization

With reference again to Fig. 3, the distance between $\tilde{g}_o(x)$ and $g_o(x)$ squared is

$$R(\vec{l},\vec{\chi}) = \int_{-\infty}^{\infty} \left| g_o(x) - \tilde{g}_o(x) \right|^2 dx$$

$$= \int_{-\infty}^{\infty} \left| \sum_{n=1}^{m} \int_{l_n}^{l_{n+1}} g_i(\xi)\left[h(x-\xi;\xi) - h(x-\xi;\chi_n)\right] d\xi \right|^2 dx \tag{25}$$

As with some of the previous measures, $R(\vec{l},\vec{\chi})$ is a function of the input plane calibration parameters here expressed in vector form as

$$\vec{l} = [l_1, l_2, \ldots, l_n, \ldots, l_{m+1}]$$
$$\vec{\chi} = [\chi_1, \chi_2, \ldots, \chi_n, \ldots, \chi_m] \tag{26}$$

under the constraint of Eq 3. A scheme for optimizing

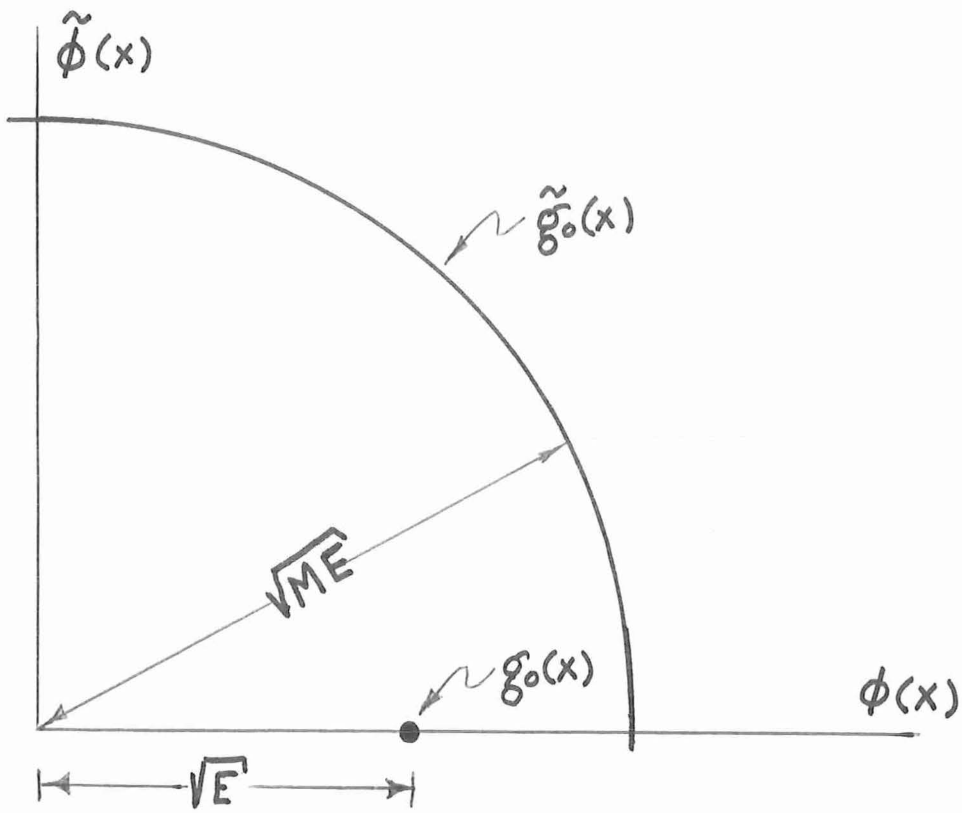


Fig 4: Illustration of the nonconvergence of the ideal magnifier's PIA output to its true output in signal space. The circular arc represents the locus of possible PIA's.

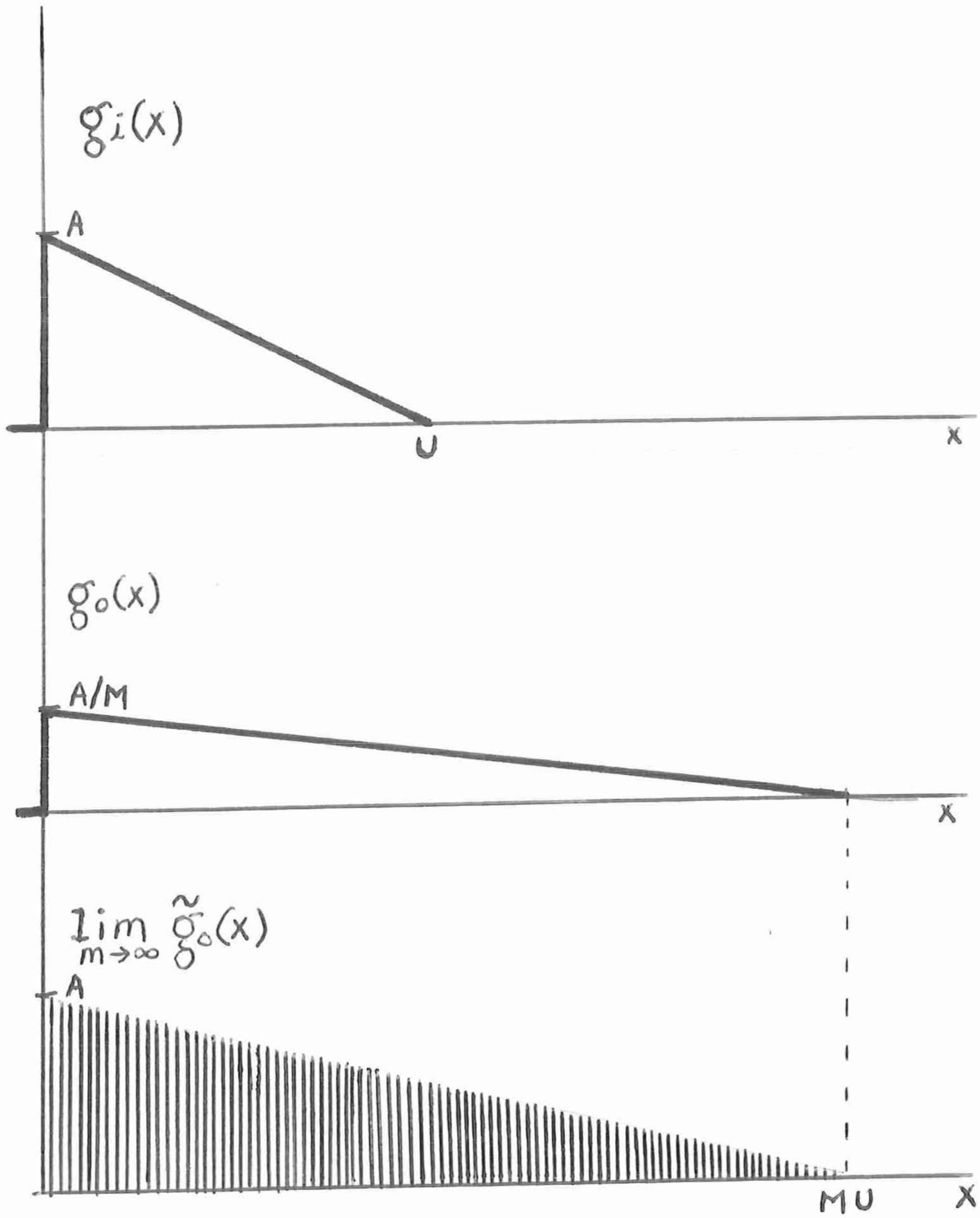


Fig 5: Illustration of the input, true output, and PIA output for a large patch density. The PIA output is seen not to converge to the true output.

of the PIA under given physical constraints is minimization of $R(\vec{1},\vec{\chi})$. This quantity may be viewed as the energy of the difference of the true and PIA outputs.

As an example of such optimization, consider the ideal magnifier with input

$$g_i(\xi) = e^{-bx}\mu(x) \quad ; \; b>0 \tag{27}$$

We assume physical constraints limit the minimun patch width to Δ . Past observation of the magnifier's PIA dictate the smaller the patch, the better the approximation. We thus assign a width of Δ to each patch and write

$$\ell_n = (n-1)\Delta \tag{28}$$

From Eq. 21 it can be shown that

$$h(x-\xi;\chi_n) = \delta\left[x-\xi-(M-1)\chi_n\right] \tag{29}$$

Substituting Eqs 27 and 29 into Eq 25 under the constraints of Eq 28, followed by simplification gives

$$R(\Delta,\vec{\chi}) = \frac{1}{2b}\left(1+\frac{1}{M}\right) - \frac{2}{b(M+1)}\left[e^{b(1+\frac{1}{M})\Delta} - 1\right] \times \sum_{n=1}^{m} e^{-bn\Delta(1+\frac{1}{M})} e^{-b(1-\frac{1}{M})\chi_n} \tag{30}$$

In that $R(\Delta,\vec{\chi})$ is positive real, and the first term is positive, we need to maximize the second term in order to minimize $R(\Delta,\vec{\chi})$. This term is maximum when χ_n is minimum. Under the constraints of Eq 3, we thus let

$$\chi_n = \ell_n = (n-1)\Delta \tag{31}$$

The result, as shown in Fig. 6 can be seen to be the best PIA of the true output under the given constraints.

C. Limitations

One should note that any signal space representation is confined to finite energy functions. That is

$$\int_{-\infty}^{\infty} |g_o(x)|^2 dx < \infty \tag{32}$$
$$\int_{-\infty}^{\infty} |\tilde{g}_o(x)|^2 dx < \infty$$

The authors have illustrated a non-finite PIA representation of a finite energy output in the case of the Fourier transformer (1).

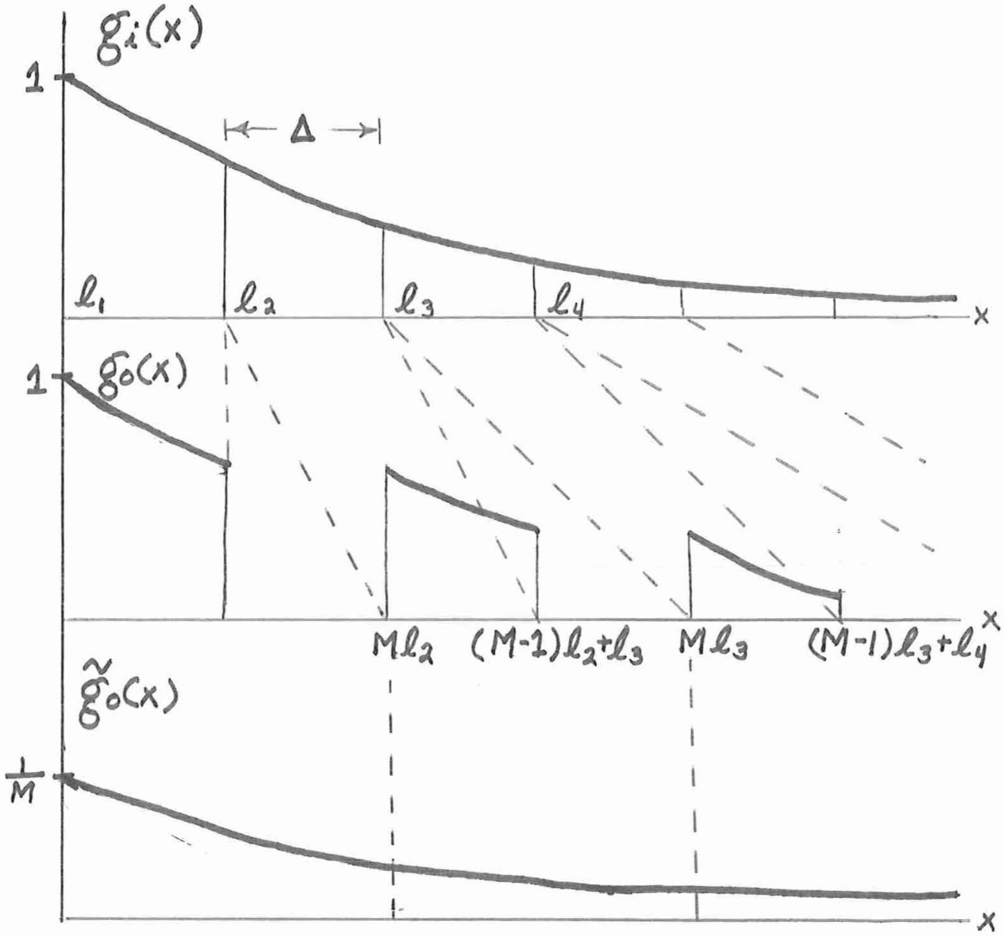


Fig 6: Optimal PIA output for the ideal magnifier with exponential input when each isoplanatic patch has width Δ.

III. Conclusions

The piecewise isoplanatic approximation (PIA), although mimicking linear space-variant system outputs, does not converge absolutely to the true output. This is due to the limited countably infinite defining relationships allowed the PIA in contrast to the unacountably infinite number of defining relationships demanded by the true output. Signal space representation of true and PIA outputs is suggested for illustrating PIA convergence and optimization for finite energy outputs.

ABSTRACT

System classifications are ranked according to the necessary transfinite number of input-output relationships required for system definition. From this consideration, the linear space-variant system output approximation through piecewise invariant modeling of the system input, termed the piecewise isoplanatic approximation (PIA), is shown not to generally converge to the true system output. Examples, employing energy comparison between true and PIA outputs, are given.

I. INTRODUCTION

The authors have presented a definitive method by which outputs of linear space-variant systems can be approximated through division of the system into a number of linear space-invariant systems.[1] The method proposed, or special cases thereof, have been successfully applied to holographic representation of the linear space-variant non-unity magnification imaging system.[2,3]

Unfortunately, in many cases of interest, the multi-space-invariant representation of the linear -variant system, termed the piecewise isoplanatic approximation(PIA), does not converge to the true output as the density of approximating invariant systems grows arbitrarily large. The underlying reason for the non-convergence lies in the differing transfinite number of input-output mapping operations capable of the PIA and required of the linear-variant system. The non-convergence of the PIA is many times exposed by comparison of the energy of the PIA and true output.

II. A SYSTEM CLASSIFICATION HIERARCHY

Systems can be classified by the number of input-output relationships required for system definition. Herein, a system is said to be defined if the system response can be predicted with knowledge of the corresponding input. A system, consisting of an input f (x), a "black box", and an output g (x), can be characterized by the operator S such that

$$g(x) = S[f(x)] \tag{1}$$

where, without loss of generality, x can be viewed as an n dimensional variable.

Consider first, the general (non-linear) case, where <u>no</u> assumptions are made concerning S. One must know the system response for all possible inputs in order to completely define the system.

A less stringent defining relationship requirement arises from the sole assumption of system linearity, the property of which may be stated as

$$S[a\,s(x) + b\,t(x)] = a\,S[s(x)] + b\,S[t(x)] \tag{2}$$

where s(x) and t(x) are arbitrary inputs and a and b are constant.

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A less stringent defining relationship requirement arises from the sole assumption of system linearity, the property of which may be stated as

$$S\left[a\,s(x) + b\,t(x)\right] = a\,S\left[s(x)\right] + b\,S\left[t(x)\right] \tag{2}$$

where s(x) and t(x) are arbitrary inputs and a and b are constant.

Such systems may be characterized by knowledge of the system response to Dirac delta inputs at each input point.[4] The impulse response corresponding to the imput point $x=\xi$ is written

$$h(x-\xi;\xi)=S[\delta(x-\xi)] \tag{3}$$

Through the properties of Eq. 2, the input-output relationship of a linear system can be shown to be defined through the superposition integral:

$$g(x)=\int_{-\infty}^{\infty} f(\xi)\,h(x-\xi;\xi)\,d\xi \tag{4}$$

Classically, the next step in developing the system classification hierarchy is assumption of the linear system's shift invariance. Shift invariant systems (not necessarily linear) are characterized by the property that the output shifts directly with the input. That is

$$g(x-\xi)=S[f(x-\xi)] \tag{5}$$

For the invariant linear (isoplanatic) system, the impulse response takes on the form

$$h(x-\xi;\xi)=h(x-\xi) \tag{6}$$

and the superposition integral of Eq. 4 becomes the convolution integral

$$g(x) = \int_{-\infty}^{\infty} f(\xi)\, h(x-\xi)\, d\xi \qquad (7)$$
$$= f(x) * h(x)$$

Note that the invariant linear system requires only one defining input-output relationship for complete system definition.

A system classification belonging between the linear and invariant linear categories is the linear piecewise invariant (LPI) system.[1,5] An LPI system is defined as a linear system whose input space is divided into disjoint invariant regions, the n^{th} of which extends over the non-zero interval defined by

$$\mu(x-\ell_n)\,\mu(\upsilon_n - x) \qquad (8)$$

where $\upsilon_n = \ell_{n+1}$ and $\mu(x)$ is the unit step function:

$$\mu(x) = \begin{cases} 0 & ;\ x < 0 \\ 1 & ;\ x \geq 0 \end{cases} \qquad (9)$$

The LPI system is completely defined through knowledge of the system responses to impulse imputs within each of the invariant regions. The input-output relationship may then be written as[1]

$$g(x) = \sum_n \int_{\ell_n}^{u_n} f(\xi)\, h_n(x-\xi)\, d\xi \qquad (10)$$
$$= \sum_n \left[f(x)\, \mu(x-\ell_n)\, \mu(u_n - x) \right] * h_n(x)$$

where

$$h_n(x - x_n) = S[\delta(x - x_n)] \qquad (11)$$

and

$$\ell_n < x_n \leq u_n \qquad (12)$$

III. NECESSARY NUMBER OF DEFINING RELATIONSHIPS

The Cantorian theory of transfinite numbers[6] consecutively orders degrees of infinity, the n^{th} of which is denoted by $\aleph_n$. Two infinite sets are equally "strong" if there exists a one to one mapping between their elements. The first few transfinite numbers and corresponding example sets of their order (strength) are

$\aleph_0$: Integers

Finite disjoint regions over a plane

$\aleph_1$: Real Numbers

Points on a plane

$\aleph_2$: All geometrical curves

Recalling the general (non-linear) system, one sees that with no knowledge of the workings within the "black box" the number of input-output relationships required for complete system definition is of order $\aleph_2$. That is, one needs to know the system output for every possible input. Similarly, the linear system requires $\aleph_1$ defining relationships since there is needed one defining relationship (impulse response) per point in the input space.[7] The LPI system, requiring one impulse response for each of the countably infinate invariant imput regions, requires $\aleph_0$ defining relationships. Finally, the linear invariant system requires only one defining relationship.

As is seen in Table I, the LPI system essentially provides a missing link in the transition between the linear and invariant linear system classifications.

SYSTEM CLASSIFICATION	REQUIRED NUMBER of DEFINING RELATIONSHIPS
General (Non-linear)	$\aleph_2$
Linear	$\aleph_1$
Linear Piecewise Invariant (Piecewise Isoplanatic)	$\aleph_0$
Linear Invariant (Isoplanatic)	1

Table 1: A system classification hierarchy with corresponding number of necessary input-output relationships for system definition.

IV. RELATIONSHIP TO THE CONVERGENCE OF THE PIECEWISE ISOPLANATIC APPROXIMATION

For purposes of holographically representing a linear space-variant systems, the authors have proposed a model by which isoplanatic variant systems might be approximated as piecewise invariant.[1,8] The linear variant system input space is divided into nonoverlapping regions within which the line spread function essentially meets the invariance criterion of Eq. 6. The resulting piecewise invariant approximation output, $\tilde{g}(x)$, is given by

$$\tilde{g}(x) = \sum_n \int_{\ell_n}^{u_n} f(\xi)\, h(x-\xi; x_n)\, d\xi \tag{13}$$

The true output, g(x), is given by the superposition integral of Eq. 4 which may be written as

$$g(x) = \sum_n \int_{\ell_n}^{u_n} f(\xi)\, h(x-\xi; \xi)\, d\xi \tag{14}$$

Comparing Eq. 13 and 14, one initially assumes that as the density of the invariantly modeled input regions grows arbitrarily large marked by a corresponding shrinkage of each invariant region, the PIA output would approach the true output. That is

$$\lim_{\substack{u_n - \ell_n \to 0 \\ n \to \infty}} \tilde{g}(x) = g(x) \tag{15}$$

Unfortunately, due to Cantorian considerations, such is not always the case. The " $n \to \infty$ " in Eq. 15 should read "$n \to \aleph_0$". That is, the PIA, in the limit, has only $\aleph_0$ possible defining relationships while g(x), being a variant system output, generally requires $\aleph_1$ defining relationships. As such, the PIA does not usually converge to the true output.

Equation 15 would be satisfied in many cases if

$$h(x;\xi) = \lim_{\substack{u_n - l_n \to 0 \\ n \to \infty}} h(x;x_n)\mu(\xi - l_n)\mu(u_n - \xi) \qquad (16)$$

The right side of Eq. 16 is a piecewise constant version of $h(x;\xi)$ in ξ , and thus, in the limit, is capable of defining $\aleph_0$ values, short of the $\aleph_1$ ordered pairs required to completely specify $h(x;\xi)$.

In many instances, the nonconvergence of the PIA may be illustrated through comparison of its energy[9] to the true output, the difference of which manifests the previous Cantorian considerations.

V. EXAMPLES

Illustration of the effects of Cantorian theory on the convergence of the PIA are now offered.

1. The Ideal One - Dimensional Imaging Systems

The impulse response for an ideal one dimensional imaging system with magnification M , is defined as

$$h(x-\xi;\xi) = \delta(x-M\xi) \tag{17}$$

One may write equivalently

$$g(x) = \frac{1}{|M|} f\left(\frac{x}{M}\right) \tag{18}$$

One sees that for $M > 1$, due to the nonconformance with Eq. 6, the system is linear and invariant.[10] The energy E contained in g(x) is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |g(x)|^2 dx \\ &= \frac{1}{M} \int_{-\infty}^{\infty} |f(\rho)|^2 d\rho \end{aligned} \tag{19}$$

The imaging system PIA, from Eqs. 17 and 13, is

$$\begin{aligned} \tilde{g}(x) = \sum_n f[x-(M-1)x_n]\,\mu[x-\ell_n-(M-1)x_n] \\ \times \mu[-x+x_n+(M-1)x_n] \end{aligned} \tag{20}$$

Note that $\tilde{g}(x)$ is merely a summation of a piecewise shifted version of $f(x)$, and lacks the $\frac{1}{M}$ amplitude scaling factor of $g\ (x)$. Nonconvergence is immediately suspected. For $M > 1$, the piecewise shifted patches are non-overlapping and the energy in the PIA output is

$$\begin{aligned} \tilde{E} &= \int_{-\infty}^{\infty} |\tilde{g}(x)|^2 dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\ &= ME \end{aligned} \tag{21}$$

Thus, the energy difference between the true output and PIA outputs for a given input, regardless of the density of the input regions, always differs by a constant. The PIA of the one dimensional imaging system then, obviously does not, in the strictest of senses converge to the true output.

In fairness, it must be said that the above energy approach is not valid for the two dimensional imaging system. ~~For $M > 1$,~~ The true and PIA outputs do indeed have equivalent energy for all $M > 1$. The PIA, however, still does not converge to the true output since it has only the power to shift $\aleph_0$ input regions, while the true output demands the shifting and amplitude scaling of $\aleph_1$ points.

2. The Thin Lens Fourier Transformer

The two dimensional thin lens Fourier transformer system has an input-output relationship of

$$g(x,y) = \int\!\!\int_{-\infty}^{\infty} f(p,q)\, e^{-j\frac{2\pi}{\lambda f}(px+qy)}\, dp\,dq = \mathfrak{F}\left[f(x,y)\right] \tag{22}$$

where f is the lens focal length and λ the wavelength of the spatially coherent illumination. The corresponding point spread function is

$$h(x-\xi, y-\eta; \xi, \eta) = e^{-j\frac{2\pi}{\lambda f}(\xi x + \eta y)} \tag{23}$$

and the corresponding PIA is

$$\tilde{g}(x,y) = \sum_n \sum_n \mathfrak{F}\Big[f(x+x_n, y+y_m) \times \mu(x+x_n-\ell_n)\,\mu(-x-x_n-u_n) \times \mu(y+y_m-\ell_m')\,\mu(-y-y_m-u_m') \Big]\Big|_{\substack{x=-x_n \\ y=-y_m}} \times e^{-j\frac{2\pi}{\lambda f}(x x_n + y y_m)} \tag{24}$$

where $\ell_n < X \leq u_n$ and $\ell'_m < Y \leq u'_m$ define the nm^{th} input region. If

$$X_n = n \Delta_X$$
$$Y_m = m \Delta_Y \tag{25}$$

then the PIA of Eq. 24 takes an a Fourier Series type of form.[1] Thus, for most inputs, the energy associated with $\tilde{g}\ (x,Y)$ is nonfinite. Through Parcval's theorem, the true output of a Fourier transformer for finite energy inputs has finite energy. It can thus be stated that the PIA of a Fourier transformer does not converge to the true output in the strictest of senses.

3. The Integrator

There do exist systems for which the PIA converges to the true output. Consider , as an example, the linear system which integrates the input over all X and displays the result as the amplitude of an output pulse. That is

$$g(x) = \mathrm{rect}\left[\frac{x}{2a}\right] \int_{-\infty}^{\infty} f(\eta)\, d\eta \qquad (26)$$

where 2 a is the output pulse width and

$$\mathrm{rect}\left[\frac{x}{2a}\right] = \mu(x + a)\,\mu(a - x) \qquad (27)$$

The corresponding line-spread function is

$$h(x - \xi ; \xi) = \mathrm{rect}\left[\frac{x}{2a}\right] \qquad (28)$$

and the system PIA output is

$$\tilde{g}(x) = \sum_n \int_{\ell_n}^{u_n} f(\xi)\ \mathrm{rect}\left[\frac{\xi - (x + x_n)}{2a}\right] d\xi \qquad (29)$$

Assuming that for all n

$$2a > u_n - \ell_n \qquad (30)$$

Equation 29 becomes

$$\begin{aligned}\tilde{g}(x) &= \sum_n \mu[x-(\ell_n - x_n - a)]\,\mu[(u_n - x_n - a) - x]\int_{\ell_n}^{x_n+x+a} f(\xi)\,d\xi \\ &+ \sum_n \mu[x-(u_n - x_n - a)]\,\mu[(\ell_n - x_n + a) - x]\int_{\ell_n}^{u_n} f(\xi)\,d\xi \\ &+ \sum_n \mu[x-(\ell_n - x_n + a)]\,\mu[(u_n - x_n + a) - x]\int_{x_n+x-a}^{u_n} f(\xi)\,d\xi \end{aligned} \tag{31}$$

If the input functions largest input isoplanatic patch has width ω_{max}, then within the output interval

$$a - \frac{w_{MAX}}{2} < x < a + \frac{w_{MAX}}{2}$$

Eq. 31's middle term gives the PIA output as

$$\begin{aligned}\tilde{g}(x) &= \sum_n \int_{\ell_n}^{u_n} f(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} f(\xi)\,d\xi \\ &= g(x) \qquad ; \left| a - \frac{w_{MAX}}{2} \right| < x \end{aligned} \tag{32}$$

As the input's isoplanatic patch density grows arbitrarily large with corresponding shrinkage of patch widths, the second term in Eq. 31 approaches the true system output. Similarly, in this limit , the first and third terms in Eq. 31 shrink into zero width intervals about the points $x = -a$ and a respectively. Thus, except at these endpoints, the PIA converges exactly to the true output.

Actually, the convergence of the PIA of the integrator to the true output should not be surprising, since the operation of integration may be defined as the limit of a sum.[11]

VI. CONCLUSION

The piecewise invariant modeling of linear variant systems, termed the piecewise isoplanatic approximation (PIA), does not in general converge to the true system output as the input isoplanatic region density grows arbitrarily large. The restriction of the PIA, in the limit, to $\aleph_0$ mapping operations opposed to the generally required $\aleph_1$ mapping operations for the variant case, is cause for this non-convergence.

The general non-exactness of the limit of the PIA and true output, however, does not greatly detract from the PIA's utility. Good output approximations have been illustrated here and elsewhere[1] for the integrator, general invariant system, ideal imaging system, and thin lens Fourier transformer. The PIA is just that, an approximation, the consequences of which should be investigated prior to application.

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2. L.M. Deen, J.F. Walkup, and M.O. Hagler. Appl. Opt. 14 (1975).

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6. G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers Open Court (1915). For a more elementary, yet enlightening discussion on transfinite numbers, see G. Gamow, One, Two, Three...Infinity, Viking Press (1962) p 14.

7. In certain optical systems, the concept of degrees of freedom may be applied to reduce the number of defining relationships from that stated for the linear variant system. Such reduction, however, employs specific geometrical constraints on the system (ie "within"the black box) and thus adds to the sole assumptions of linearity and invariance considered here. See G.T. diFrancia, J.Opt.Soc.Am., 59 ,799 (1969).

8. The PIA, conceptually, has previously been suggested by Goodman (ref 4) as well as P.B. Fellgett and E.H. Linfoot, Roy.Soc.Phil.Trans.,A-247,369 (1955)

9. Energy, as used herein, refers to the integral of the signal modulus squared generalized to any dimension. See J.M. Wozencraft and I.M. Jacobs, Principles of Communication Engineering, John Wiley & Sons, 1965, p 238

10. There seems to exist a rhetorical conflict on the linear system classification of the ideal imaging system. Goodman (ref 4, pgs 19,95) claims invariance while, for example, A.A. Sawchuk, J.Opt.Soc.Am. 64 138 (1974) concludes it "rigourously" a space-variant system. In this and previous papers (ref. 1,3) the authors adopt the latter view, maintaining the invariant classification is more mathématical than physical.

11. The limit summation definition of the integral may be found in any good Advaned Calculus text. For example, see J.M.H. Olmsted, Advanced Calculus, Appleton-Century-Crofts, 1961, p.110.

Inadequacies of the Lohmann-Paris Measure of Space Invariance for Non-Imaging Systems

Robert J. Marks II
Dept. of Electrical Engineering
Texas Tech University
Lubbock, Texas 79409
(Dec. 1976)

I. Introduction: The Lohmann-Paris Invariance Measure

In order to employ such mathematical niceties as Fourier analysis, attention to linear space-variant system input planes is often times confined to an isoplanatic patch within which the system is somewhat space-invariant [1]. Summing the effects of adjacent isoplanatic patches yields an approximate variant system output [2]. As such, a measure of the "invariance" of a space-variant system is desirable.

Seemingly motivated by the definition of the complex degree of coherence encountered in statistical optics, Lohmann and Paris [3] have offered such a measure for the specific case of space-variant imaging systems. Their proposed "degree of invariance" measure of a space-variant imaging system is a normalized cross-correlation between shifted line-spread functions originating from impulse inputs at different locations on the input plane. For the one-dimensional case:

$$\sigma(p,q) = \frac{\int_{-\infty}^{\infty} h(x;p)h^*(x;q)\, dx}{\left[\int_{-\infty}^{\infty} |h(x;p)|^2\, dx \int_{-\infty}^{\infty} |h(x;q)|^2\, dx \right]^{1/2}} \tag{1}$$

where $\sigma(p,q)$ is the linear system's degree of invariance, p and q are two points on the input plane, and $h(x - \xi;\xi)$ is the linear system's response to an input $\delta(x - \xi)$ where $\delta(x)$ is the Dirac delta [4]. Due to Schwarz's inequality:

$$0 \leq |\sigma(p,q)| \leq 1 \tag{2}$$

For the space-invariant case, the line-spread function shifts directly with the input impulse. Thus

$$h(x - \xi;\xi) = h(x - \xi) \tag{3}$$

substituting the above invariant case into Eq.(1) yields a degree of invariance of unity. Conversely, a linear imaging system with a degree of invariance of zero would be said to have no trace of invariance.

Also proposed by Lohmann and Paris is an analytic definition of the isoplanatic patch. If ε is the maximum "variance" allowed a patch, then the patch interval from p to q must satisfy the following inequality:

$$| 1 - \sigma(p,q) | \leq \varepsilon \tag{4}$$

Although intuitive in design, the Lohmann-Paris measure of invariance seems limited to imaging systems. Application of this measure to certain non-imaging space-variant systems, as will be shown, yields results in direct conflict with the theory's intent.

II. <u>Inadequacies</u>

The following applications show that the Lohmann-Paris measure is inadequate for measuring the spatial invariance of certain non-imaging space-variant systems.

A. The isoplanatic patch constraint

The proposed isoplanatic patch constraint can assign the same variance to isoplanatic patches of grossly varying widths. Consider the equality version of the isoplanatic patch constraint of Eq.(4). It is possible to have a solution of this relationship of the form

$$\varepsilon = |1 - \sigma(p_o, q_o)| = |1 - \sigma(p_o, q_o + \Delta q)| \tag{5}$$

where p_o and q_o are points of the input plane and Δq is a non-zero interval. Assuming $p_o < q_o$ and $\Delta q > 0$, the interval from p_o to q_o is assigned the same invariance as from p_o to $q_o + \Delta q$.

As a specific example, consider the ideal magnifier with magnification M whose inputs are frequency limited to the interval $f_x \leq |W|$. The space-variant magnifier's line-spread function is

$$h(x - \xi;\xi) = \delta(x - M\xi) \tag{6}$$

Through Parseval's theorem, the frequency domain equivalent of Eq(1) is

$$\sigma(p,q) = \frac{\int_{-\infty}^{\infty} H_x(f_x;p)H_x^*(f_x;q)\, df_x}{[\int_{-\infty}^{\infty} |H_x(f_x;p)|^2\, df_x \int_{-\infty}^{\infty} |H_x(f_x;q)|^2\, df_x]} \tag{7}$$

where $H(f_x;\xi)$ is the Fourier transform of $h(x;\xi)$:

$$\begin{aligned} H_x(f_x;\xi) &= \int_{-\infty}^{\infty} h(x;\xi)\exp(-j2\pi f_x x)\, dx \\ &= \mathcal{F}_x[h(x;\xi)] \end{aligned} \tag{8}$$

Substituting Eq(6) into Eq.(8) yields

$$\begin{aligned} H_x(f_x;\xi) &= [\delta\{x - (M - 1)\xi\}] \\ &= \exp[-j2\pi(M - 1)\xi f_x] \end{aligned} \tag{9}$$

and with the cited frequency constraints substituted into Eq.(7) gives the magnifier's degree of invariance [5] as

$$\sigma(p,q) = \text{sinc}[2(M - 1)(p - q)W] \tag{10}$$

where

$$\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

The Lohmann-Paris isoplanatic patch constraints for the ideal magnifier is thus

$$|1 - \text{sinc}[2(M - 1)(p - q)W]| \leq \varepsilon \tag{11}$$

An illustration of Eq.(11) for a typical ε is offered in Fig.(1). The constraint is satisfied in region 1 and 3, but not in region 2 even though the interval $p - q$ is smaller than in region 3. This is in direct conflict with the general observation that the samller the patch, the greater the invariance.

B. Failure to predict piecewise isoplanatic modeling

The Lohmann-Paris invariance measure can erroneously predict the unsuccessful piecewise isoplanatic modeling of a space variant system [2].

Consider the degree of invariance of the ideal magnifier without frequency constraint. (i.e. let $W \rightarrow \infty$):

$$\sigma(p,q) = \lim_{W\to\infty} \text{sinc}[2(M - 1)(p - q)W]$$

$$= \begin{cases} 1 \ ; \ p = q \\ 0 \ ; \ p \neq q \end{cases} \tag{12}$$

A similar degree of invariance is assigned to the ideal thin lens Fourier transformer [5] with the line-spread function

$$h(x - \xi;\xi) = \exp\left(-j\frac{2\pi}{\lambda f}x\right) \tag{13}$$

where λ is the wavelength of the coherent illumination and f is the focal length of the lens. If the inputs are space limited to the interval (-a,a), the degree of invariance of the Fourier transformer is

$$\sigma(p,q) = \mathrm{sinc}\left[\frac{2(p - q)a}{\lambda f}\right]\exp\left[-j\frac{\pi}{\lambda f}(p^2 - q^2)\right] \tag{14}$$

The degree of invariance for the unrestricted Fourier transformer is then

$$\sigma(p,q) = \lim_{a\to\infty} \mathrm{sinc}\left[\frac{2(p - q)a}{\lambda f}\right]\exp\left[-j\frac{\pi}{\lambda f}(p^2 - q^2)\right]$$

$$= \begin{cases} 1 \ ; \ p = q \\ 0 \ ; \ p \neq q \end{cases} \tag{15}$$

Both the ideal magnifier ($M \neq 1$) and the ideal thin lens Fourier transformer are thus predicted to have no trace of invariance. This implies no successful piecewise isoplanatic modeling of these systems can be made. This is contrary to successful results of such piecewise isoplanatic modeling previously presented [2]. The frequency limited magnifier can be characterized exactly by a sampling theorem approach [6].

C. "Quasi-linear"system description

A "Quasi-linear" system, as defined by Arsenault and Brousseau [7], is a system which is space-invariant only for a set class of inputs. These authors have noted that the Lohmann-Paris method may yield a space variant measure for such systems.

Conversely, there exist quasi-linear systems which are assigned total invariance by the Lohmann-Paris measure. Consider the quasi-linear piecewise isoplanatic system with line-spread function

$$h(x - \xi;\xi) = \mathrm{rect}\left(\frac{x - \xi}{2a}\right)\mathrm{rect}\left(\frac{\xi}{2b}\right) \tag{16}$$

where

$$\mathrm{rect}(x) = \begin{cases} 1 \ ; & |x| \leq 1/2 \\ 0 \ ; & \text{otherwise} \end{cases} \tag{17}$$

and a and b are constants. Such a system is invariant for object inputs which are zero outside the interval $-b \leq \xi \leq b$. Substituting into Eq.(1) gives the degree of invariance of this system as

$$\sigma(p,q) = 1 \tag{18}$$

Here is a case where a linear system does not meet the classical invariance criterion of Eq.(3), yet is classified as totally invariant based on the Lohmann-Paris measure.

D. Separable line-spread functions

As a final example, consider the case where the line-spread function is separable. That is

$$h(x;\xi) = f(x)g(\xi) \tag{19}$$

Such a system could be viewed as an invariant system with line-spread function $f(x - \xi)$ with a transmittance $g(\xi)$ placed in its input plane. The line-spread function in Eq.(16) describes such a system. If the transmittance $g(\xi)$ is positive and real, then the predicted degree of invariance is

$$\sigma(p,q) = 1 \tag{20}$$

Based on the measure of Eq.(1), total invariance is again predicted for a system which is classically space-variant.

III. Conclusions

The Lohmann-Paris measure of the degree of invariance previously applied to space-variant imaging systems has been shown inadequate in the following more general applications:

1. Definition of the isoplanatic patch (for the frequency limited magnifier with $M \neq 1$).
2. Predicting successful piecewise isoplanatic modeling of certain space-variant systems (such as the Fourier transformer).
3. Measuring the degree of invariance of quasi-linear systems.
4. Measuring the degree of invariance for linear space-variant systems with separable line-spread functions.

These inadequacies dictate the need for a revised or augmented measure of the invariance of linear non-imaging space-variant systems.

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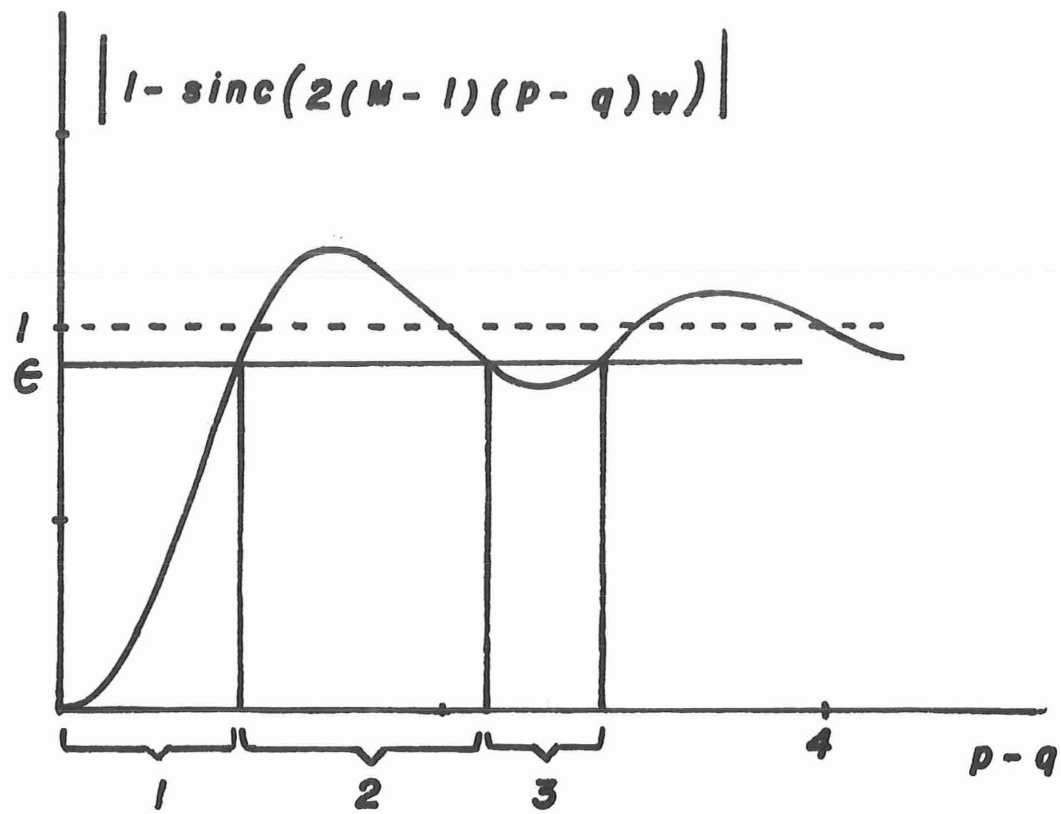
Figure Caprions

Fig. 1 : Shown is the degree of invariance of an ideal magnifier as a function of input patch width, p - q. By the given definition of the isoplanatic patch, patche widths corresponding to region 1 are allowable. Patch widths in region 3 are also allowable even though they are larger than patch widths in region 2. This violates the observation that the larger the patch, the less the degree of invariance.


|1- sinc(2(N- 1)(p- q)w)|
1
ε
1
2
3
4
p - q

SERIES ITEM
AND GENERAL
SAMPLING

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

27 June 1978

Professor Thomas F. Krile
Rose-Hulman Institute of Technology
5500 Wabash Avenue
Terre Haute, Indiana 47803

Dear Tom,

I really enjoyed our discussions and fellowship at the Gordon Conference. We'll do it all over again in October in San Francisco at the OSA Meeting.

Enclosed are copies of some correspondence I've been having with Gary Wise on "Laplace series." I'm hoping it's a good method for inverse Laplace transformation.

The foundation of the Laplace series comes from Szasz's theorem:

Let x(t) be causal, Lebegue measureable and square integratable:

$$\int_0^\infty |x(t)|^2 \, dt < \infty .$$

We then say that $x(t) \in L_2[0,\infty) = L_2^+$. The function set, $\{e^{-b_n t} \mid 0 \le t < \infty, n = 1,2,3,\ldots\}$ is complete in L_2^+ iff

$$\operatorname{Re} b_n > 0 \qquad \text{for all } n$$

and

$$\sum_{n=1}^\infty \frac{\operatorname{Re} b_n}{1 + |b_n - \tfrac{1}{2}|^2} = \infty .$$

In English, this means that we can completely characterize x(t) with knowledge of the inner product of x(t) with each of the basis functions. But the inner product of x(t) with $e^{-b_n t}$ is a sample of x(t)'s Laplace transform:

$$\int_0^\infty x(t)\, e^{-b_n t} \, dt = X(b_n)$$

where

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt .$$

Thus, upon choosing an applicable sequence $\{b_n \mid n = 1,2,3,\ldots\}$, we can (in principle) form an interpolation set, $\{\psi_n(t) \mid n = 1,2,3,\ldots\}$ determined solely by the b_n's such that for every $x(t) \in L_2^+$ we can write

$$x(t) = \sum_{n=1}^{\infty} X(b_n)\, \psi_n(t) .$$

The equality here is rigourously in the L_2 norm. That is,

$$\lim_{N\to\infty} \left[\int_0^\infty \left| x(t) - \sum_{n=1}^{N} X(b_n)\, \psi_n(t) \right|^2 dt \right]^{\frac{1}{2}} = 0 .$$

The two enclosed memos derive the interpolation function for the case where $b_n = (n + \frac{1}{2})r$. The constant r is assumed positive (but is otherwise arbitrary) and parameterizes the sampling rate in the Laplace domain. It turns out in this case that the interpolation function takes on the form

$$\psi_n(t) = rI_n(rt) ,$$

where $I_n(t)$ is a weighted sum of "distorted" Legendre polynomials.

Hope this of help to you!

Best wishes,

Robert J. Marks II
Assistant Professor

P.S.: I'd really appreciate it if you could send me some references on the other methods of inverse Laplace transformation that you mentioned. Some references on the undergraduate lab fiber optics experiments we talked about would also be most welcome.

RM:bb
enclosures

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enclosures

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

28 June 1978

Professor Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Gary,

Got an application! Inverse Laplace transformation. I talked to Tom Krile at the Gordon Conference, and he mentioned there were no "good" digital methods. Maybe the Laplace series is the answer.

Here's an extention of the Laplace series which allows sampling to begin at points other than $r/2$ on the real axis in the s-plane:

Theorem

Let $x(t) \in L_2^+$. Then, for every $\tau > 0$, $r > 0$, and complex number a where $\operatorname{Re} a > 0$, we have the relation:

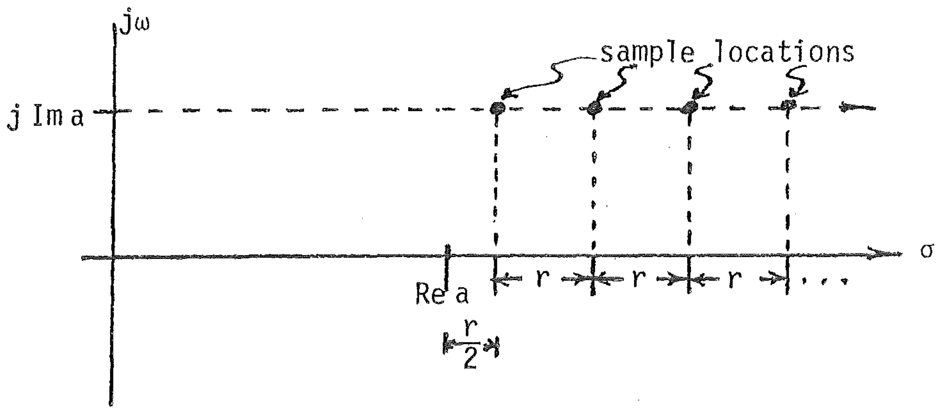
$$x(t) = re^{at}\, e^{-(a+\frac{1}{2}r)\tau} \sum_{n=1}^{\infty} X[r(n+\tfrac{1}{2}) + a]\, e^{-nr\tau}\, I_n[r(t+\tau)] \ ,$$

where the expression for $I_n(t)$ was given in my previous letter to you (16 June 78) and $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt = \mathscr{L}[x(t)] \ .$$

Equality is in the L_2 norm.

Before proving the theorem, let me give some personality to some of the variables. We are sampling $X(s)$ in the s-plane as follows:



As you can see, Re a parameterizes where our sampling begins. The parameter r specifies our sampling interval parallel to the σ-axis. The imaginary component of a dictates the distance away from the σ-axis we sample. In most applications, I would imagine we would set $\text{Im}\, a = 0$. The parameter τ is not pictured in the figure. We apparently loose nothing by setting $\tau = 0$. τ, however, might play a role in the series convergence rate.

Here's the theorem proof. We begin with the expression developed in the last correspondence:

$$x(t) = r \sum_{n=1}^{\infty} X[(n+\tfrac{1}{2})r]\, I_n(rt) \quad ; \quad x(t) \in L_2^+,\ r > 0 \ .$$

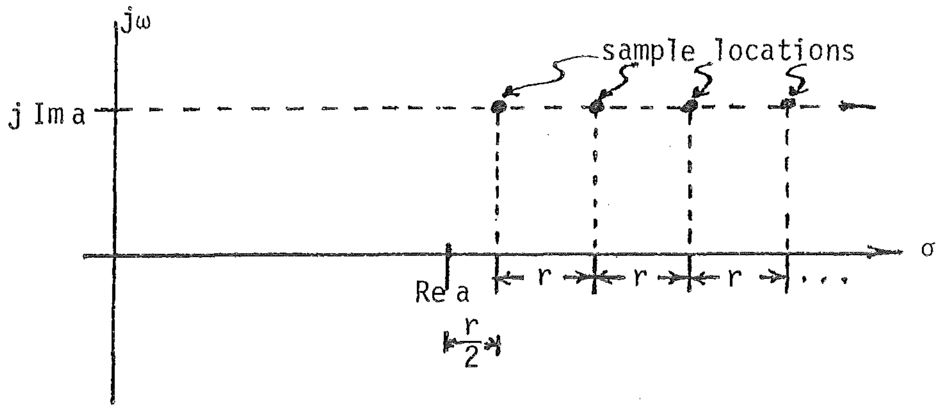
For every $x(t) \in L_2^+$ and every $\tau > 0$ and for every a with the property $\text{Re}\, a \geq 0$, it follows that

$$x(t-\tau)\, e^{-at} \ \in \ L_2^+ \ .$$

This statement follows straightforwardly from Schwarz's inequality. We now make use of the Laplace transform relation:

$$\mathcal{L}[x(t-\tau)\, e^{-at}] = X[s+a]\, e^{-(a+s)\tau} \ .$$

Using the Laplace series, we can write:



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Using the Laplace series, we can write:

Professor Gary Wise
28 June 197[illegible]
Page 3

$$x(t-\tau)e^{at} = r \sum_{n=1}^{\infty} X[r(n+\tfrac{1}{2}) + a]\, e^{-[r(n+\frac{1}{2})+a]\tau}\, I_n(rt) .$$

A straightforward manipulation followed by a shift of t to $t+\tau$ completes the proof.

These results have yet to be digitally verified. Only time and an IBM 370 will tell.

Best wishes,

Bob

Robert J. Marks II
Assistant Professor

RM:bb

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

19 September 1978

Dr. Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Dear Gary,

Summer's over and I'm looking forward to another year of fun. Working here at the Applied Physics Lab was enjoyable this summer, but I like independent research much better.

I admit that the 1943 paper you sent me sunk my boat a little bit. I agree with your comment, though, that there may still be good things to discover.

In the Legendre polynomial treatment, Mike Hall and I have come up with some convergence problems I'd like to share with you.

Let's first backtrack to the June 16 letter to you in which it was shown that, if $x(t) \in L_2^+$ and r is a positive constant, then, with equality in the L_2 norm, we have

$$x(t) = \sum_{n=1}^{\infty} \alpha_n \phi_n(t) ,$$

where the orthonormal basis function is

$$\phi_n(t) = [r(2n-1)]^{\frac{1}{2}} e^{-(rt/2)} P_{n-1}[2e^{-rt} - 1] \; ; \; n = 1,2,3...$$

and the inner product is

$$\alpha_n = \frac{[r(2n-1)]^{\frac{1}{2}}}{(-2)^{n-1}} \sum_{q=0}^{n-1} \frac{(-2)^q}{q!} X[r(q+\tfrac{1}{2})] \, C_{nq} \; . \tag{1}$$

Here, X(s) refers to the Laplace transform of x(t) and

$$C_{n+1,q} = \sum_{k=o}^{[\frac{n-q}{2}]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k-q)!} .$$

It turns out that C_{nq} can be put in a better form. Recall that

$$P_n(t) = 2^{-n} \sum_{k=o}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} t^{n-2k} .$$

From this, we can show that

$$C_{n+1,q} = 2^n \left(\frac{d}{dt}\right)^q P_n(1)$$

which, in turn, can be shown to be

$$C_{n+1,q} = \frac{2^{n-q}(n+q)!}{q!(n-q)!} .$$

Substituting into (1) gives

$$\alpha_n = (-1)^{n+1} [r(2n-1)]^{\frac{1}{2}} \sum_{q=o}^{n-1} \frac{(-1)^q (n+q-1)!}{(q!)^2 (n-q-1)!} X[r(q+\tfrac{1}{2})] .$$

We know from Parseval's theorem that

$$\int_0^\infty |x(t)|^2 \, dt = \sum_{n=1}^{\infty} |\alpha_n|^2 .$$

A necessary condition for the series on the right to converge is that α_n must tend to zero as n gets large.

Let's take a typical L_2^+ signal:

$$x(t) = e^{-t} , \qquad t > 0 .$$

Then, for $r = 1$, we have

$$X[q+\tfrac{1}{2}] = (q + \frac{3}{2})^{-1} .$$

The corresponding α_n's are

$$\alpha_n = (-1)^{n+1} \; [2n-1]^{\frac{1}{2}} \sum_{q=0}^{n-1} \frac{(-1)^q (n+q-1)!}{(q!)^2 (n-q-1)!(q+\frac{3}{2})} .$$

They do not go to zero as n goes to infinity. We've racked our brains and can't figure out why.

Any ideas?

Best wishes,

Bob

Robert J. Marks II
Assistant Professor

P.S.: Will have a draft of the Laplace computer program paper to you this month.

RM:bb

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

10 October 1978

Dr. Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Gary,

Here's something interesting:

Let u(x) be real and in $L_2[-\infty,\infty]$. Define the linear transform:

$$g(s) = \frac{1}{j2\pi} \int_{-\infty}^{\infty} \frac{u(x)\ dx}{x - \frac{js}{2\pi}} .$$

Then the sample values, $g(\lambda_n)$, are sufficient to characterize u(s) if the sample locations, λ_n; $n=1,2,3...$, satisfy Szasz's criterion. In other words, the set

$$\frac{1}{j2\pi x + \lambda_n}$$

forms a basis set for all real $L_2(-\infty,\infty)$ signals.

Furthermore, if u(z), $z = x + jy$, is the analytic continuation of u(x), u(z) is analytic on the upper half-plane, and

$$\lim_{y\to\infty} u(z) = 0 .$$

Then, from Cauchy's integral theorem:

$$g(s) = u(\frac{js}{2\pi}) .$$

Then, if all λ_n's are real (eg., $\lambda_n = n$), then u(z) is characterized by sampling u(z) along the positive imaginary axis! (Neat!)

Proof:

If $u(t) \in L_2(-\infty,\infty)$ is real, then its Fourier transform is Hermetian:

$$U(f) = U^*(-f) ,$$

where

$$U(f) = \int_{-\infty}^{\infty} u(t)\ e^{-j2\pi ft}\ dt .$$

Also, $U(f) \in L_2(-\infty,\infty)$, which implies that $U(f)\mu(f) \in L_2^+$, where $\mu(\cdot)$ is the unit step function.

From Szasz's theorem, $U(f)\mu(f)$ can be specified from the inner product

$$a_n = \int_0^\infty U(f)\, e^{-\lambda_n f}\, df ,$$

where the λ_n's satisfy Szasz's criterion. Then,

$$a_n = \int_0^\infty \left[\int_{-\infty}^\infty u(x)\, e^{-j2\pi fx}\, dx\right] e^{-\lambda_n f}\, df$$

$$= \int_{-\infty}^\infty u(x) \int_0^\infty e^{-(\lambda_n + j2\pi x)f}\, df dx$$

$$= \frac{1}{j2\pi} \int_{-\infty}^\infty \frac{u(x)\, dx}{x - \frac{j\lambda_n}{2\pi}} .$$

The statement concerning analyticity of u clearly follows from Cauchy's integral theorem and the fact that $\mathcal{Re}\,\lambda_n > 0$.

Q.E.D.

Let $I_n(f)$ be the unique interpolation function such that

$$U(f)\mu(f) = \sum_{n=1}^\infty a_n\, I_n(f)\, \mu(f) .$$

[This is the function of Legendre polynomials with which we're having convergence challenges for $\lambda_n = n$.] Using the fact that

$$U(f) = U(f)\mu(f) + U^*(-f)\mu(-f) ,$$

it is easy to show that:

$$u(x) = 2\,\mathcal{Re} \sum_{n=1}^\infty a_n\, \phi_n(x) ,$$

where

$$\phi_n(x) = \int_0^\infty I_n(f)\, e^{j2\pi fx}\, df .$$

Let me know what you think.

Best wishes,

Robert J. Marks II
Assistant Professor

RM:bb

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

23 October 1978

Dr. Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Dear Gary,

We have solved the convergence problem! It turns out that the interpolation formula:

$$x(t) = re^{-rt/2} \sum_{n=o}^{\infty} (-1)^n (2n+1)\, P_n[2e^{-rt} - 1] \sum_{q=o}^{n} \frac{(-1)^q}{(q!)^2} \frac{(n+q)!}{(n-q)!} X[r(q+\tfrac{1}{2})] \tag{1}$$

is correct. We cannot, however, make the change of summation order. That is,

$$\sum_{n=o}^{\infty} \sum_{q=o}^{n} \neq \sum_{q=o}^{\infty} \sum_{n=q}^{\infty} .$$

This is due to the fact that, in general, the series is not absolutely convergent. That is,

$$\sum_{n=o}^{\infty} |b_n(t)| = \infty ,$$

where the b_n's are everthing inside the n summation sign in (1). [Can we find a x(t) for which it <u>does</u> uniformly converge?]

There <u>must</u>, however, exist a unique interpolation formula, $I_q(t)$, such that

$$x(t) = \sum_{q=o}^{\infty} X[r(q+\tfrac{1}{2})]\, I_q(t) \tag{2}$$

It turns out, though, that we can't find $I_q(t)$ by switching the sum sign in (1). It must be computed in a different manner. Any ideas?

Let's see what we can salvage from (1). In any "practical" situation, our ∞ index will be finite. Call it N. In this case, we can write

$$\sum_{n=o}^{N} \sum_{q=o}^{n} = \sum_{q=o}^{N} \sum_{n=q}^{N} .$$

Then (1) becomes

$$x(t) \simeq r \sum_{q=o}^{N} X[r(q+\tfrac{1}{2})] \; I_q^{(N)}(rt) \;, \tag{3}$$

where

$$I_q^{(N)}(t) = e^{-t/2} \frac{(-1)^q}{(q!)^2} \sum_{n=q}^{N} (-1)^n (2n+1) \frac{(n+q)!}{(n-q)!} P_n[2e^{-t} - 1] \; .$$

The relation in (3) would seem to be applicable to a good approximation. Analytically, we could find x(t) from

$$x(t) = r \lim_{N\to\infty} \sum_{q=o}^{N} X[r(q+\tfrac{1}{2})] \; I_q^{(N)}(rt) \; . \tag{4}$$

By our previous observation, this is <u>not</u> equivalent to

$$r \sum_{q=o}^{\infty} X[r(q+\tfrac{1}{2})] \lim_{N\to\infty} I_q^{(N)}(rt \; .$$

We're gonna be playing around with this some more from two different angles:

1. Trying to find $I_q(t)$ in (2) via a different approach.
2. Looking at some digital implementations of (3).

By the way, the α_n's I said diverged in my letter of 19 September 1978 actually converged. The actual problem was that the proposed $I_q(t)$'s were <u>not</u> in L_2^+. Now we know why.

Best wishes,

Bob

Robert J. Marks II
Assistant Professor

RM:bb

Gary
Here's an ~~exciting~~ fresh view of the inverse Laplace transform series.
We depict the L_2^+ Hilbert space by the Szasz basis elements:

$$\{e^{-r(q+\frac{1}{2})t}\mu(t) \mid q=0,1,2,\ldots\ ;\ r>0\} \qquad (1)$$

~~Denote the N~~
Let A_N be an N dimensional ~~finite~~ subspace of L_2^+ ~~subspace~~ defined as that space ~~defined~~ spanned by the ~~two~~ N basis elements:

$$\{e^{-r(q+\frac{1}{2})t}\mu(t) \mid q=0,1,2,\ldots,N\ ;\ r>0\} \qquad (2)$$

Here's the practical case of the digital inverse Laplace transform: We are given ~~a finite number the Lap~~ the inner product of $x(t)\in L_2^+$ with each of the ~~N~~ ~~the exp~~ above exponential basis elements in A_N. These inner products, of course, are ~~are the sampled~~ Laplace transform samples:

$$\{X[r(q+\tfrac{1}{2})] \mid q=0,1,2,\ldots,N\ ;\ r>0\} \qquad (3)$$

where

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt \qquad (4)$$

The question is: How do we best estimate $x(t)$ by a linear series expansion using the basis elements in (2) and the inner products in (3)? First, we must

~~reo~~ recognize that any such estimate will ~~be~~ lie ~~lie~~ in the A_N space. Thus, the best estimate we can make of the signal is the projection of $x(t)$ in L_2^+ onto the A_N subspace. This fact follows from the projection theorem. By "best", we mean that the L_2 norm (distance) between $x(t)$ and all ~~elem~~ points in A_N is minimized.

The best estimate of $x(t)$ [call it $\tilde{x}_N(t)$] ~~is~~ is thus obtained by a conventional series approach. The result, from the 23 Oct 78 memo, is

$$\tilde{x}_N(t) = r \sum_{q=0}^{N} X[r(q+\tfrac{1}{2})] \; I_q^{(N)}(rt) \qquad (5)$$

where

$$\cancel{I_q^{(N)}(t) = e^{-t/2}\frac{(-1)^q}{q!}}$$

$$I_q^{(N)}(t) = \frac{(-1)^q}{(q!)^2} e^{-\frac{t}{2}} \sum_{n=q}^{N} (-1)^n (2n+1) \frac{(n+q)!}{(n-q)!} P_n[2e^{-t}-1] \qquad (6)$$

We conclude ~~that~~ for all $y_N(t) \in A_N$, the estimate $\tilde{x}_N(t) = y_N(t)$ minimizes the L_2^+ norm (mean square error):

$$\left[\int_0^\infty |x(t) - y_N(t)|^2 dt\right]^{1/2}$$

This result is almost trivially simple but nonetheless significant.

Some comments:

1) Let $\tilde{X}_N(s)$ denote the Laplace transform of $\tilde{x}_N(t)$:

~~$\tilde{x}_N(t) = \int_0^\infty$.~~

$$\tilde{X}_N(s) = \int_0^\infty \tilde{x}_N(t)\, e^{-st}\, dt$$

Then

$$\tilde{X}_N[r(q+\tfrac{1}{2})] = X[r(q+\tfrac{1}{2})]\ ;\ q = 0, 1, 2, \ldots, N$$

This follows from the fact that both $x(t)$ and $\tilde{x}_N(t)$ both have the same coordinates in the first N dimensions of L_2^+ space.

2) Nowhere is it required that our sampling rate index, r, ~~is~~ be real. If complex, we require ~~that~~ only that

$$\operatorname{Re} r > 0$$

This allows inverse Laplace transform estimation given the appropriate samples in the right half of the complex s plane along any line passing through the origin. Mean square convergence is again insured.

~~3) Even with the above result, we must take our first sample at $s = r/2$. Here's a generalization that allows us to begin sampling at $p + r/2$ as long as Re p is positive. From comment #1, we can write:~~

3) Suppose we had the Laplace transform samples:

$$\{\mathbb{X}[r(q+\tfrac{1}{2})] \mid q=M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r>0\}$$

The best estimate of $x(t)$ here is

$$\tilde{x}_{N-M}(t) = r \sum_{q=M}^{N} \mathbb{X}[r(q+\tfrac{1}{2})]\, I_q^{(N)}(rt)$$

It's easy to show that this expansion uses only ~~has as the~~ basis elements

$\cancel{\{e^{-r}}$

$$\{e^{-r(q+\frac{1}{2})}\mu(t) \mid q=M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r>0\}$$

to span the M-N dimensional subspace of L_2^+.

4) Even with the above, we are required to take our first sample at $r(M+\frac{1}{2})$. This restricts our sampling rate. If we wish to take the first sample at $r(M+\frac{1}{2})+a$ where $0 \leq \mathrm{Re}\, a \leq \frac{1}{2}\mathrm{Re}\, r$, the corresponding best estimate is

$$\tilde{x}(t) = r\, e^{at} \sum_{q=M}^{N} \mathbb{X}[r(q+\tfrac{1}{2})+a]\, I_q^{(N)}(rt)$$

The Szasz basis elements here are

$$\cancel{\{e^{-r(q+\frac{1}{2})+a} \mid q=M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r>0, \mathrm{Re}\, a \leq \tfrac{1}{2}\mathrm{Re}\, r}$$

$$\cancel{\{e^{-r(q+\frac{1}{2})+a} \mid q=M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r>0, 0 \leq \mathrm{Re}\, a \leq \tfrac{1}{2}\mathrm{Re}\, r}$$

$$\{e^{-r(q+\frac{1}{2})+a} \mid q=M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r>0,\ 0 \leq \mathrm{Re}\, a \leq \tfrac{1}{2}\mathrm{Re}\, r\}$$

This expansion follows from the fact that

$$\mathcal{L}[\tilde{x}(t)e^{-at}] = X(s+a)$$

~~Here~~

Here's the whole thing in a nutshell: Let $x(t) \in L_2^+$ have a Laplace transform $X(s)$. Given only $N-M$ ~~the Laplace transform samples:~~ ~~samples~~ regularly spaced ~~$\{X[r(q+\frac{1}{2})\}$~~ samples of $X(s)$ ~~in the right~~ along any line in the right half plane:

$$\{X[r(q+\tfrac{1}{2}) + a];\ q = M, M+1, \ldots, N-1, N;\ \mathrm{Re}\, r > 0,\ 0 \le \mathrm{Re}\, a \le \tfrac{1}{2}\mathrm{Re}\, r\}$$

the estimation of $x(t)$ ~~which but~~ with minimum norm in L_2^+ space is

$$\tilde{x}(t) = r e^{at} \sum_{q=M}^{N} X[r(q+\tfrac{1}{2}) + a]\ I_q^{(N)}(rt)$$

where

~~$I_q^{(N)}(t) =$~~

$$I_q^{(N)}(t) = \frac{(-1)^q}{(q!)^2} e^{-t/2} \sum_{n=q}^{N} (-1)^n (2n+1) \frac{(n+q)!}{(n-q)!} P_n[2e^{-t} - 1]$$

Gary,

Enjoyed your Legendre paper.

Gotta couple of thoughts to share. Let r be a positive constant. Then $e^{-rt/2} P_n[1-2e^{-rt}]$ is complete on L_2^+. Since (from your Legendre paper) there exists a set $\{h_{nm}\}$ such that

$$e^{j\pi n t} = \sum_m h_{nm} P_m(t)$$

it follows that

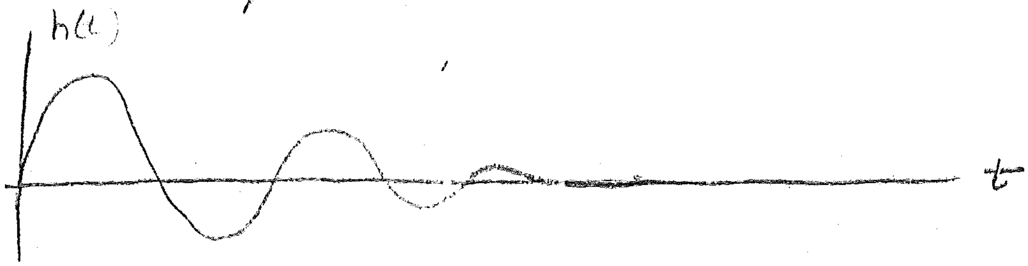
$$e^{-rt/2} e^{j\pi n(1-2e^{-rt})} = \sum_m h_{nm}\left[e^{-\frac{rt}{2}} P_m(1-2e^{-rt})\right]$$

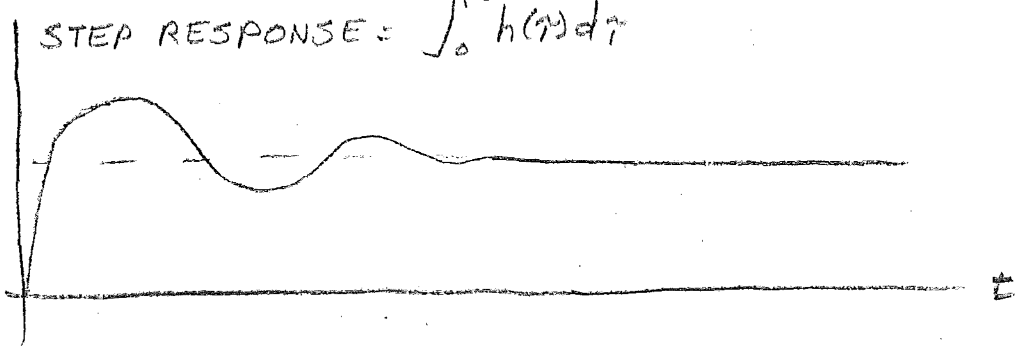
is also complete on L_2^+. In fact

(1) $$\psi_n(t) = \sqrt{r}\,(-1)^n e^{-rt/2} e^{-j2\pi n e^{-rt}}$$

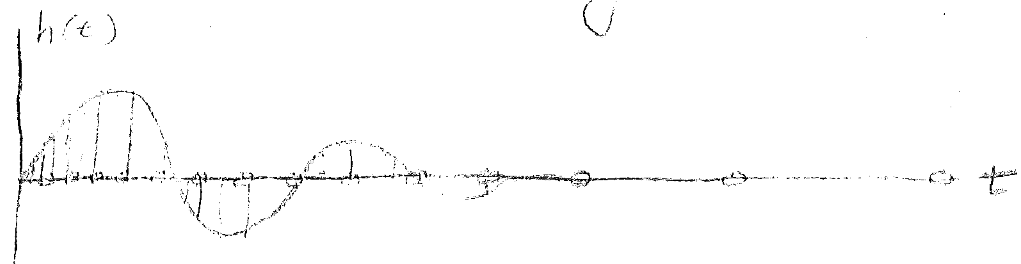
is a complete orthonormal basis set on L_2^+.

Now for an application. Suppose we have a signal with high frequency components near the origin which dies to zero. A prime example is the classic impulse response:





More "information" is near the origin. It then seems we would want to take more samples there. Contrarily, near steady state, we would not need as many samples. They just give redundant information. Our desired sampling would look something like this:



So much for a preliminary justification.

Now, suppose from the sample values we want to establish the coefficients for expanding $h(t)$ in terms of $\psi_n(t)$ in ①. We thus form the inner product

$$\alpha_n = \int_0^\infty h(t)\,\sqrt{r}\,(-1)^n e^{-rt/2} e^{-j2\pi n e^{-rt}}\,dt$$

Making the variable substitution

$$\hat{t} = 1 - 2e^{-rt}$$

gives

$$\alpha_n = \int_{-1}^{1} \frac{h\left[-\frac{1}{r}\ln\frac{1-\hat{t}}{2}\right]}{\sqrt{2r(1-\hat{t})}}\, e^{-j2\pi n\hat{t}}\,d\hat{t}$$

The α_n's can be easily computed by a FFT. The FFT must be fed uniformly spaced samples.

We are essentially finding the Fourier series of

$$y(\hat{t}) = \frac{h[\phi(\hat{t})]}{\sqrt{2r(1-\hat{t})}} \quad \text{on } [-1, 1] \text{ where}$$

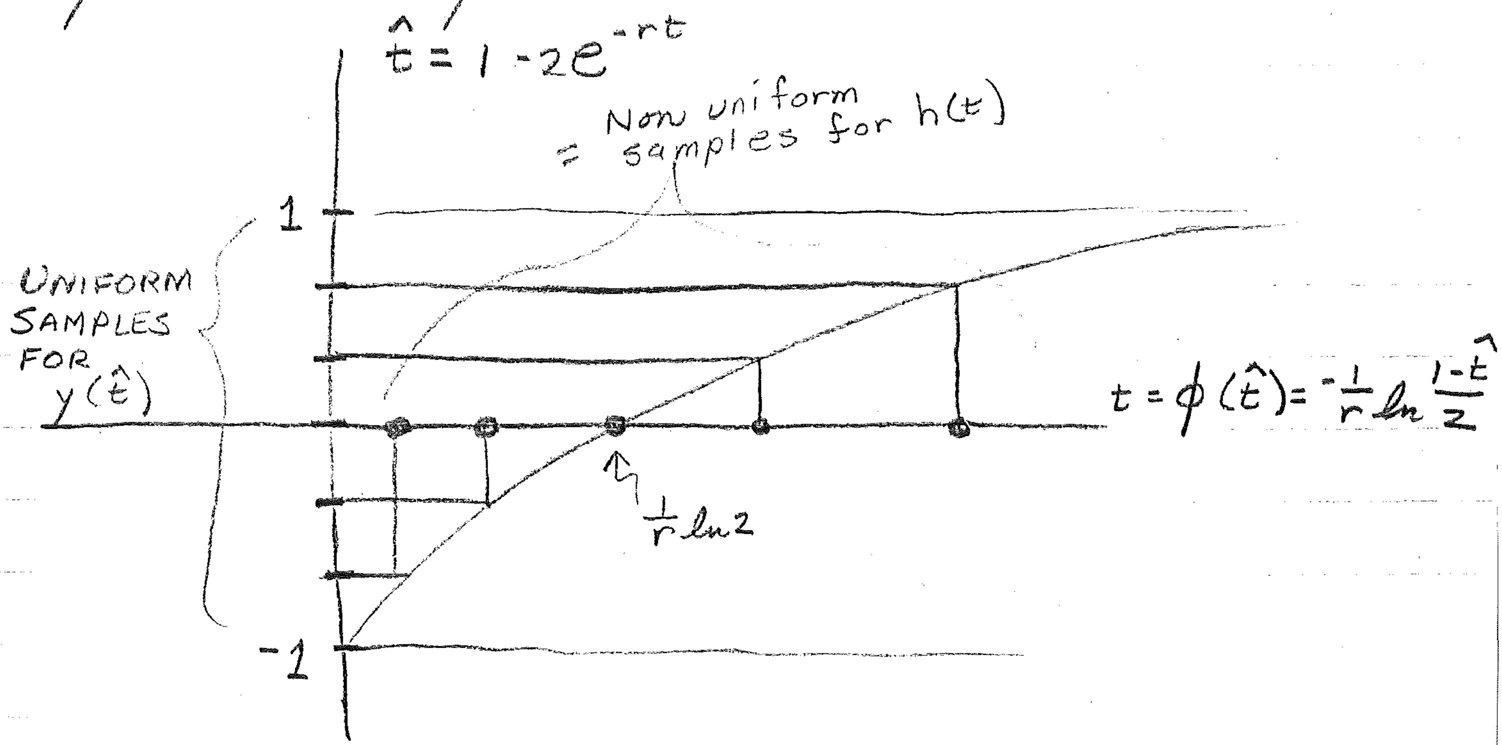
$$\phi(\hat{t}) = -\frac{1}{r} \ln \frac{1-\hat{t}}{2}$$

We give the FFT the sample values $y(\frac{p}{M})$, $p = 0, \pm 1, \pm 2, \dots, \pm M$ where $2M+1$ = total number of samples. But

$$y\left(\frac{p}{M}\right) = \frac{h[\phi(p/M)]}{\sqrt{2r(1-\frac{p}{M})}} \Leftarrow$$

Note: Must make $y(1)=0$. ie, we reach the steady state for $h(t)$ and $h[\phi(1)] = 0$

$h[\phi(\frac{p}{M})]$ is simply our nonuniformly spaced samples!



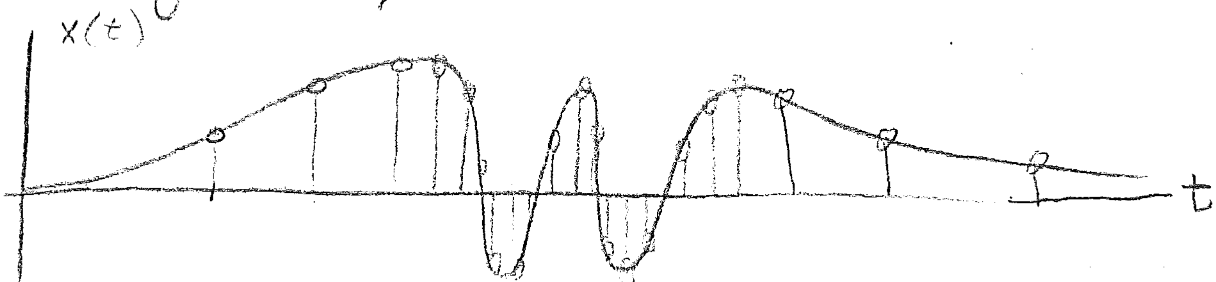
Thus, we can sample $h(t)$ at a higher rate at the origin (where there is more info) than in steady state (where there is little info). Treating these sampes as uniform, we find the Fourier series coefficients, α_n, for $y(t)$. These are the same coefficients which are the expansion coefficients using $\psi_n(t)$ in ① as a basis. If we wished to use $c_n e^{-rt/2} P_n[1-2e^{-rt}]$ as a basis, just whip out your Legendre matrix from your paper and turn the crank. I think that's neat! We're economizing our data!

End of phase 1.

Gary,

Here's phase 2.

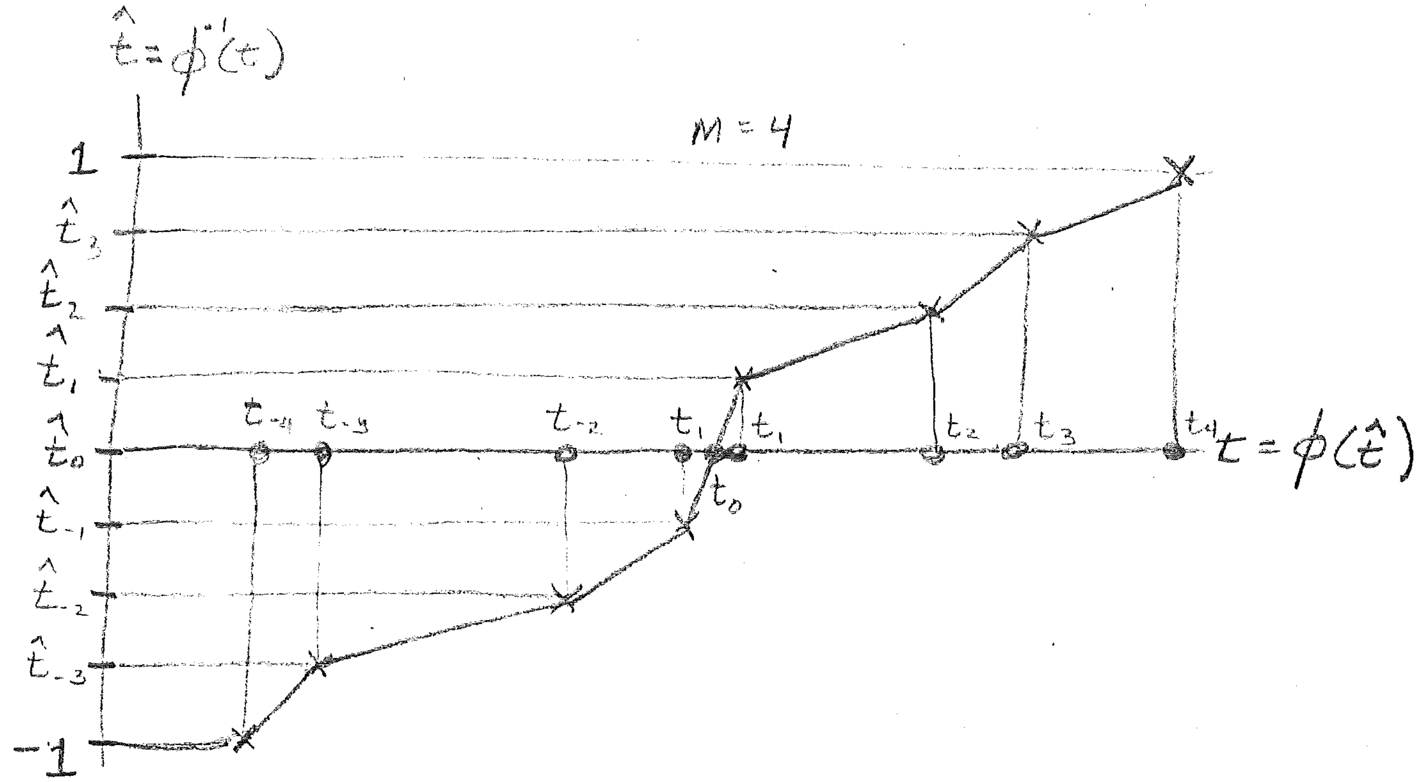
Seems to me that the logical extension of the last correspondance is a scheme wherein the sampling rate is a function of the signal. I apeal to the concept of a signal's "instantaneous frequency." When the instantaneous frequency is high, we sample faster. I offer the following example:



Question is: Given the location of the n^{th} sample point from the origin and the signal's value there, how can we approximate the original signal?

Here's a possible scheme. See what you think ⟹

First, form the piecewise linear function*
$\hat{t} = \phi^{-1}(t)$ from the sample time locations.
$\phi^{-1}(t)$ maps the, say, $2M+1$ sample
locations onto $[-1, 1]$ For example:



[On the last letter, we were looking at the special case $\hat{t} = \phi^{-1}(t) = 1 - 2e^{-rt}$.]
Montonicity of $\phi^{-1}(t) = \hat{t}$ is obvious.
Thus, we have an inverse, $t = \phi(\hat{t})$.

* Think a least squares "smooth distortion" would be better. Use at least quadratic fit.

Next, we form the function

$$y(\hat{t}) = \frac{d\phi}{d\hat{t}}\, x[\phi(\hat{t})]$$

where $\frac{d\phi}{d\hat{t}}$ is the transformation Jacobian. We feed uniform sampled values of $y(\hat{t})$ [which are simply deterministically weighted versions of the nonuniformly sampled $x(t)$] into a FFT to find the α_p's where

$$y(\hat{t}) \simeq \sum_{p=-M}^{M} \alpha_p\, e^{-j\pi p t}$$

Note we must compute $\left.\frac{d\phi}{d\hat{t}}\right|_{\text{SAMPLE POINT}} = \phi'_p$. This could be estimated in a number of ways.

Anyway, we regain $x(t)$ from

$$x(t) = \frac{d\phi^{-1}}{dt}\, y[\phi^{-1}(t)]$$

$$= \frac{d\phi^{-1}}{dt} \sum_{p=-M}^{M} \alpha_p\, e^{-j\pi p \phi^{-1}(t)}$$

If you wanna use Legendre series, find coefficient B_q from the α_p's with your matrix. Then

$$x(t) = \frac{d\phi^{-1}}{dt} \sum_q B_q P_q[\phi^{-1}(t)]$$

Whadaya think?

NOTE ON AN INVERSION FORMULA FOR THE LAPLACE TRANSFORMATION

A. ERDÉLYI*.

1. A sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ of real or complex numbers will be called a *base* for the Laplace transformation if any Laplace integral

$$g(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{1}$$

vanishing at all points $s = \bar{\lambda}_m$ ($m = 0, 1, 2, \ldots$: $\bar{\lambda}$ is the conjugate complex to λ) necessarily vanishes identically. Such bases exist: thus any sequence $\{\lambda_m\}$ with a finite point of condensation is obviously a base, and there are also bases without a finite point of condensation. For example, a celebrated theorem of Lerch states that $\{s_0+m\}$ is such a base. If $\{\lambda_m\}$ is a base for the Laplace transformation, then both $f(t)$ and $g(s)$ itself (the former except for an additive null-function) are determined uniquely by the values the latter function takes at the points $s = \bar{\lambda}_m$.

The expression of $f(t)$ in terms of $g(\bar{\lambda}_m)$ ($m = 0, 1, 2, \ldots$) is a new inversion formula for the Laplace transformation. Of the usual inversion formulae, the so-called complex inversion requires the knowledge of $g(s)$ along a line parallel to the imaginary axis; the Paley-Wiener and the Boas-Widder inversion formulae assume the knowledge of $g(s)$ along the real axis; the inversion by means of Laguerre polynomials makes use of the values of $g(s)$ and all its derivatives at a finite point; and the so-called Post-Widder inversion formula involves the values of $g(s)$ and all its derivatives for large positive real s. It is perhaps of some interest to have an inversion formula which requires the knowledge of $g(s)$ only at a certain denumerable set. From the practical point of view, the new inversion is likely to be useful in cases when $g(s)$ is determined by numerical methods, for instance in certain cases of the application of the Heaviside calculus.

The expression of $g(s)$ in terms of $g(\bar{\lambda}_m)$ yields an interpolation formula. This new interpolation formula is slightly reminiscent of the cardinal series and its generalisations, but the resemblance is not as close as perhaps at first it might appear. Though the new interpolation applies to a certain class of functions analytic in a half-plane, and so is more general than

* Received 17 January, 1943; read 18 March, 1943.

the interpolation by mean
certain classes of integral fu
of, and is simpler than, the
In the present note I
functions $f(t)$ and the corre
half-plane. The investigati
which are generalisations of
I hope to give a more deta
classes of functions, the c
expansions, the so-called La

the Stieltjes transformation

and certain representation t

Th

2. Suppose that all the
the real part of each of th
functions $\{e^{-\lambda_m t}\}$. These fu
quadratically integrable ove
orthonormal set $\{\phi_n\}$ of func

(2) ϕ

i.e. a set for which

$$\int_0^\infty \phi_m \bar{\phi}_n\, dt = 0$$

The well-known method
yields an explicit expression

(3) $\phi_n(t)$

where D_n denotes the dete
k-th column is $(\bar{\lambda}_i+\lambda_k)^{-1}$ (i, k

the interpolation by means of the cardinal series (which interpolates certain classes of integral functions), yet the latter is not a particular case of, and is simpler than, the former.

In the present note I restrict myself to quadratically integrable functions $f(t)$ and the corresponding class of functions $g(s)$ analytic in a half-plane. The investigation is based on an orthonormal set of functions which are generalisations of the Jacobi polynomials. In a future paper I hope to give a more detailed account of the subject, including other classes of functions, the convergence and summability theory of the expansions, the so-called Laplace-Stieltjes transformation

$$g(s) = \int_0^\infty e^{-st}\, da(t),$$

the Stieltjes transformation

$$g(s) = \int_0^\infty \frac{da(t)}{s+t},$$

and certain representation theorems.

The orthonormal system.

2. Suppose that all the λ_m are different from one another and that the real part of each of them is positive, and consider the sequence of functions $\{e^{-\lambda_m t}\}$. These functions are linearly independent and each is quadratically integrable over $(0, \infty)$: hence it is possible to determine an orthonormal set $\{\phi_n\}$ of functions

$$\phi_n(t) = \sum_{m=0}^{n} c_{mn} e^{-\lambda_m t}, \tag{2}$$

i.e. a set for which

$$\int_0^\infty \phi_m \bar{\phi}_n\, dt = 0 \ \text{ if } \ m \neq n \quad \text{and} \quad = 1 \ \text{ if } \ m = n.$$

The well-known method of orthogonalisation of Gram and Schmidt yields an explicit expression for ϕ_n, viz.

$$\phi_n(t) = c_n (D_n \bar{D}_{n-1})^{-\frac{1}{2}} D_n(t), \tag{3}$$

where D_n denotes the determinant whose element in the i-th row and k-th column is $(\bar{\lambda}_i + \lambda_k)^{-1}$ $(i, k = 0, 1, \ldots, n)$, $D_n(t)$ the determinant obtained by replacing $(\bar{\lambda}_n+\lambda_k)^{-1}$ by $e^{-\lambda_k t}$ $(k=0, 1, \ldots, n)$ in D_n, and c_n is an arbitrary complex constant of modulus one.

Now D_n and the cofactors of $e^{-\lambda_0 t}, \ldots, e^{-\lambda_n t}$ in $D_n(t)$ are double alternants and can easily be evaluated [cf. *e.g.* (3), § 353]. With a suitable choice of c_n we obtain

$$c_{mn} = (\lambda_n+\bar{\lambda}_n)^{\frac{1}{2}} \prod_{i=0}^{n-1} (\lambda_m+\bar{\lambda}_i) \Big/ \prod_{k=0}^{n}{}' (\lambda_m-\lambda_k), \tag{4}$$

where the prime at the product-sign indicates omission of the vanishing factor $k=m$.

The sequence of functions thus determined is orthonormal in $L_2(0, \infty)$, for any choice of the sequence $\{\lambda_m\}$, provided only that no two λ's are equal and that $\Re(\lambda_m)>0$ for $m=0, 1, 2, \ldots$. From an important result due to Szász we deduce, by a simple change of variable, that $\{e^{-\lambda_m t}\}$, and therefore also $\{\phi_n\}$, *is complete with respect to and closed in* $L_2(0, \infty)$ *if and only if the infinite series*

$$\Sigma \frac{\Re(\lambda_n)}{1+|\lambda_n|^2}$$

is divergent. In the sequel we assume that this condition is satisfied.

3. In order to abbreviate the formulae, we introduce two sets of operators $\{\Gamma_n\}$ and $\{\Gamma_n{}^*\}$, operating on functions of σ, which are defined for $\sigma=\lambda_0, \lambda_1, \lambda_2, \ldots$ and $\sigma=\bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2, \ldots$ respectively. The operators are defined by the equations

$$\Gamma_n[g(\sigma)] = \sum_{m=0}^{n} c_{mn}\, g(\lambda_m), \quad \Gamma_n{}^*[g(\sigma)] = \sum_{m=0}^{n} \bar{c}_{mn}\, g(\bar{\lambda}_m) \quad (n=0, 1, 2, \ldots). \tag{5}$$

Obviously $\overline{\Gamma_n[g]} = \Gamma_n{}^*[\bar{g}]$ and in particular

$$\Gamma_n[e^{-\sigma t}] = \phi_n(t), \quad \Gamma_n{}^*[e^{-\sigma t}] = \bar{\phi}_n(t). \tag{6}$$

The inversion and interpolation formulae.

4. Suppose that $f(t)$ and $g(s)$ are connected by (1). The Fourier expansion of $f(t)$ associated with $\{\phi_n\}$ can be written in the form

$$\sum_{n=0}^{\infty} \Gamma_n[e^{-\sigma t}] \int_0^{\infty} f(t)\, \Gamma_n{}^*[e^{-\sigma t}]\, dt.$$

Now

$$\int_0^\infty f(t)\,\Gamma_n{}^*[e^{-\sigma t}]\,dt = \Gamma_n{}^*\left[\int_0^\infty f(t)\,e^{-\sigma t}\,dt\right] = \Gamma_n{}^*[g(\sigma)],$$

and hence we find the new form

$$\sum_{n=0}^{\infty} \Gamma_n[e^{-\sigma t}]\,\Gamma_n{}^*[g(\sigma)] \tag{7}$$

for the Fourier expansion of $f(t)$. Under suitable conditions on $\{\lambda_m\}$ and $f(t)$, or $g(s)$, this series is convergent or in some sense summable to $f(t)$, and then we have our inversion formula.

Substituting the expansion (7) in (1) for $f(t)$, and integrating term by term, we find the expansion for $g(s)$, viz.

$$\sum_{n=0}^{\infty} \int_0^\infty e^{-st}\,\Gamma_n[e^{-\sigma t}]\,dt\,\Gamma_n{}^*[g(\sigma)];$$

and since

$$\int_0^\infty e^{-st}\,\Gamma_n[e^{-\sigma t}]\,dt = \Gamma_n\left[\int_0^\infty e^{-(s+\sigma)t}\,dt\right] = \Gamma_n[(s+\sigma)^{-1}],$$

this expansion can be written as

$$\sum_{n=0}^{\infty} \Gamma_n[(s+\sigma)^{-1}]\,\Gamma_n{}^*[g(\sigma)]. \tag{8}$$

Again we expect that, if suitable conditions are imposed upon $\{\lambda_m\}$ and $g(s)$, the expansion (8) associated with $g(s)$ will converge or be in some sense summable to this function, and thus furnish us with an interpolation formula for Laplace transforms, *i.e.* for functions representable by a Laplace integral (1).

5. We give three simple theorems concerning the expansions (7) and (8).

THEOREM 1. *If $f(t)$ belongs to $L_2(0, \infty)$ then $g(s)$ is defined for $\Re(s) > 0$, the partial sums of (7) converge in mean square over $(0, \infty)$ to $f(t)$, and the partial sums of (8) converge to $g(s)$ for $\Re(s) > 0$.*

The proof is simple. The existence of $g(s)$ for $\Re(s) > 0$ is obvious. The convergence in mean of the partial sums of (7) to $f(t)$ follows from the closure property of $\{\phi_n\}$. Finally, since (7) converges in mean square and e^{-st} belongs to $L_2(0, \infty)$, the convergence of (8) may be deduced by term-by-term integration.

In order to formulate our second theorem, we define† $\mathfrak{H}_2{}^*$ as the class of functions $g(s)$ analytic for $\Re(s) > 0$ and such that

$$\int_{-\infty}^{\infty} |g(\xi+i\eta)|^2 \, d\eta \leqslant M^2$$

for all $\xi > 0$.

Theorem 2. *If $g(s)$ belongs to $\mathfrak{H}_2{}^*$ then the partial sums of* (8) *converge to $g(s)$ for $\Re(s) > 0$; the partial sums of* (7) *converge, in mean square over* $(0, \infty)$, *to a function $f(t)$ of $L_2(0, \infty)$; and* (1) *holds, with this $f(t)$, for* $\Re(s) > 0$.

For, since $g(s)$ belongs to $\mathfrak{H}_2{}^*$, there is a function $f(t)$ of $L_2(0, \infty)$ such that (1) holds for $\Re(s) > 0$ [cf. (1), Satz 1]; and we can apply Theorem 1 to this function.

Let us denote the N-th partial sums of (7) and (8) by $f_N(t)$ and $g_N(s)$. Then $g(s) - g_N(s)$ is the Laplace transform of $f(t) - f_N(t)$ and the latter function belongs to $L_2(0, \infty)$. It follows from Parseval's formula for Fourier transforms, and the mean square convergence of $f_N(t)$ to $f(t)$ in $(0, \infty)$, that $g_N(s)$ converges to $g(s)$ in mean square over $(-i\infty, i\infty)$. Further, if $\xi > 0$, then $e^{-\xi t} f(t) - e^{-\xi t} f_N(t)$ belongs to $L_p(0, \infty)$ for $1 \leqslant p \leqslant 2$, and $e^{-\xi t} f_N(t)$ converges to $e^{-\xi t} f(t)$ in p-th mean over $(0, \infty)$. An inequality due to Titchmarsh‡ then shows that $g_N(s)$ converges to $g(s)$ in p'-th mean, where $p' = p/(p-1)$, over $(\xi - i\infty, \xi + i\infty)$. We thus have

Theorem 3. *If $g(s)$ belongs to $\mathfrak{H}_2{}^*$, then the partial sums of* (8) *converge to $g(s)$ in mean square over $(-i\infty, i\infty)$. If $\xi > 0$, then they converge to $g(s)$ in mean, with any index not less than* 2, *over* $(\xi - i\infty, \xi + i\infty)$.

6. We conclude with a few remarks on the sequence of functions

$$\psi_n(s) = (2\pi)^{-\frac{1}{2}} \Gamma_n[(s+\sigma)^{-1}] = (2\pi)^{-\frac{1}{2}} \sum_{m=0}^{n} \frac{c_{mn}}{s+\lambda_m}. \tag{9}$$

A metric can be introduced in $\mathfrak{H}_2{}^*$ by the definition

$$\|g(s)\| = \lim_{\xi \to +0} \int_{-\infty}^{\infty} |g(\xi+i\eta)|^2 \, d\eta.$$

From Theorem 3 it follows at once that $\{\psi_n\}$ is closed in, and therefore complete with respect to, the metric space defined in this way. Moreover

† Following Doetsch (1).

‡ Cf. (6), p. 96, formula (4.1.2).

$\{\psi_n\}$ is an orthonormal system; for $\psi_n(s)$ is the Laplace transform of $(2\pi)^{-\frac{1}{2}}\phi_n(t)$, and the orthogonal property of ψ_n over $(-i\infty, i\infty)$ follows, by Parseval's theorem, from that of ϕ_n over $(0, \infty)$. Indeed (8) is the Fourier expansion of $g(s)$ with respect to the orthonormal system $\{\psi_n\}$. For, since $g(s)$ belongs to $\mathfrak{H}_2{}^*$, it can be represented by the Cauchy integral

$$g(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(i\eta)}{s-i\eta}\,d\eta \quad [\Re(s) > 0].$$

Hence

$$\Gamma_n{}^*[g(\sigma)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(i\eta)\,\Gamma_n{}^*[(\sigma-i\eta)^{-1}]\,d\eta = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} g(i\eta)\,\overline{\psi_n(i\eta)}\,d\eta,$$

and thus (8) may be written

$$\sum_{n=0}^{\infty} \psi_n(s) \int_{-\infty}^{\infty} g(i\eta)\,\overline{\psi_n(i\eta)}\,d\eta.$$

The convergence of the expansion to $g(s)$, in mean square over $(-i\infty, i\infty)$, now follows from Theorem 3.

The formulae representing ϕ_n and ψ_n simplify considerably for particular sequences $\{\lambda_m\}$. The most interesting case is that of *equidistant* λ's. In this case ϕ_n can be expressed in terms of Jacobi polynomials and ψ_n in terms of the generalised hypergeometric function ${}_3F_2$ of unit argument. In the limiting case when all the λ_m become equal, the ϕ_n reduce to Laguerre's orthonormal system and the ψ_n to powers of a linear function of s: this case has been discussed by many authors, and recently by Shohat, who gives references to earlier literature.

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The Mathematical Institute,
The University,
Edinburgh, 1.

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NUMERICAL INVERSION OF THE LAPLACE TRANSFORM BY USE OF JACOBI POLYNOMIALS*

MAX K. MILLER† AND W. T. GUY, JR.‡

Abstract. Functional values of a function f are determined from the values $F(s)$ of its Laplace transform at discrete points of s. Evaluation of $F(s)$ at points given by $s = (\beta + 1 + k)\delta$, $k = 0, 1, \cdots$, determine coefficients in an infinite series expansion of $f(t)$ in terms of Jacobi polynomials. The values of β and δ determine the position along the real s-axis at which $F(s)$ is evaluated. An approximation to $f(t)$ is given by using a finite number of terms of the infinite series expansion of $f(t)$. Numerical examples are given and results are compared with some known numerical methods for approximating $f(t)$.

Introduction. The problem of numerically inverting the Laplace transform is known to mathematicians, physicists, and engineers and has been discussed extensively in the mathematical literature [1]–[8]. A single method for numerically inverting the Laplace transform that works equally well for all types of problems encountered is lacking. In many practical problems where the Laplace transform can be evaluated at discrete points along the real axis of the independent variables, the method described here is useful. This method is fast (economical) on the digital computers now available, and it has the advantage that for only a few computations the unknown inverse can be approximated over a large range of values in the t domain.

The Laplace transform of $f(t)$ is defined by the integral

$$F(s) = \int_0^\infty \exp(-st) f(t)\, dt, \quad \operatorname{Re} s \geqq c > 0. \tag{1}$$

For purposes of discussion here it will be assumed that the integral in (1) exists for $\operatorname{Re} s > 0$. A suitable translation of the imaginary axis can be made if this is not the case, and the theory developed here is still applicable.

The inverse Laplace transform is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(st)\, F(s)\, ds, \tag{2}$$

provided that the integral in (2) converges absolutely for $\operatorname{Re} s > c$, c sufficiently large.

Change of variable. Consider the Laplace transform of $f(t)$ defined by (1) and assume that $F(s)$ is known or can be computed at discrete points along

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† Texas Instruments, Incorporated, Dallas, Texas.
‡ Department of Mathematics, University of Texas, Austin, Texas.

the real s-axis. The variable of integration may be changed by the substitution

$$x = 2 \exp(-\delta t) - 1, \tag{3}$$

where δ is a real positive number. It follows that

$$\exp(-st) = (1 + x/2)^{s/\delta}.$$

If this equation is solved for t, then

$$t = -(1/\delta) \log [(1 + x)/2]$$

and a new function g is defined over $(-1, 1)$ by

$$g(x) = f\{-(1/\delta) \log [(1 + x)/2]\} = f(t). \tag{4}$$

In order to extend the domain of definition for g, define $g(1)$ and $g(-1)$ by

$$g(1) = \lim_{x \to 1^-} g(x),$$

and

$$g(-1) = \lim_{x \to -1^+} g(x).$$

Essentially these definitions require that $f(0) = \lim_{t \to 0^+} f(t)$ and $f(\infty) = \lim_{t \to \infty} f(t)$ be finite. If f is continuous, then g is also a continuous function. Substitution of (3) into (1) and some algebraic manipulation give

$$F(s) = (1/2\delta) \int_{-1}^{1} (1 + x/2)^{(s/\delta - 1)} g(x) \, dx. \tag{5}$$

Assume that g can be expanded over $[-1, 1]$ in an infinite series of orthogonal polynomials. The Jacobi polynomials form such a set over $[-1, 1]$. The normalized Jacobi polynomial of degree n is defined by [9]

$$P_n^{(0,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\beta} \frac{d^n}{dx^n} [(1 - x)^n (1 + x)^{n+\beta}], \tag{6}$$

where the parameter α which usually appears in this definition is zero and $\beta > -1$. For $n = 0$, $P_n^{(0,\beta)}(x) = 1$. If g can be expanded over $[-1, 1]$ in terms of the Jacobi polynomials, then

$$g(x) = \sum_{n=0}^{\infty} C_n P_n^{(0,\beta)}(x). \tag{7}$$

If the coefficients C_n are known, then $g(x)$ is known, which implies that $f(t)$ can be calculated for any $t = t_0$ by means of (4).

Insertion of the previous series into the integral in (5) yields

$$F(s) = (1/2\delta) \int_{-1}^{1} (1 + x/2)^{(s/\delta - 1)} \left[\sum_{n=0}^{\infty} C_n \, P_n^{(0,\beta)}(x) \right] dx.$$

By substituting $s = (\beta + 1 + k)\delta$ into the previous equation and simplifying terms one has

$$(8) \qquad \delta F[(\beta + 1 + k)\delta] = 2^{(\beta + k + 1)} \int_{-1}^{1} (1 + x)^{\beta + k} \left[\sum_{n=0}^{\infty} C_n \, P_n^{(0,\beta)}(x) \right] dx.$$

The factor $(1 + x)^k$ which appears in (8) may be expressed as a finite linear combination of Jacobi polynomials. That is, $(1 + x)^k$ is given by

$$(9) \qquad (1 + x)^k = a_0 P_0^{(0,\beta)}(x) + a_1 P_1^{(0,\beta)}(x) + \cdots + a_k P_k^{(0,\beta)}(x).$$

For $0 \leqq m \leqq k$, a typical coefficient a_m is a function of k and β. In order to evaluate a_m, multiply both sides of (9) by $(1 + x)^{\beta} P_m^{(0,\beta)}(x)$ and integrate over $[-1, 1]$. Because of the orthogonality property of the Jacobi polynomials, there is only one nonzero term on the right, and therefore,

$$(10) \qquad \int_{-1}^{1} (1 + x)^k (1 + x)^{\beta} P_m^{(0,\beta)}(x) \, dx = a_m \frac{2^{\beta+1}}{2m + \beta + 1}.$$

The factor $(2^{\beta+1})/(2m + \beta + 1)$ on the right is the normalization term for the Jacobi polynomials. Let it be denoted by h_m.

The Jacobi polynomial $P_m^{(0,\beta)}(x)$ can be expressed in the form

$$P_m^{(0,\beta)}(x) = b_0 + b_1(1 + x) + \cdots + b_m(1 + x)^m,$$

where the b's can be determined. However, this is not necessary. Substitution of $P_m^{(0,\beta)}(x)$ in this form into the previous integral gives

$$(11) \qquad \begin{aligned} a_m h_m &= \int_{-1}^{1} (1 + x)^{k+\beta} [b_0 + b_1(1 + x) + \cdots + b_m(1 + x)^m] \, dx \\ &= b_0 \frac{2^{k+\beta+1}}{k + \beta + 1} + b_1 \frac{2^{k+\beta+2}}{k + \beta + 2} + \cdots + b_m \frac{2^{k+\beta+m+1}}{k + \beta + m + 1}. \end{aligned}$$

If the unknown a_m is considered as a function of the parameter k, then one may write

$$(12) \quad a_m h_m = \frac{Q_m(k)}{[k + (\beta + 1)]\,[k + (\beta + 2)] \cdots [k + (\beta + m + 1)]}.$$

$Q_m(k)$ is a polynomial in the symbol "k" of degree m. The explicit expression for $Q_m(k)$ may be determined by the use of (9) and (10). In (10) let $k = m - 1$ and because of the orthogonality of the Jacobi polynomials,

$$\int_{-1}^{1} (1 + x)^{m-1} (1 + x)^{\beta} P_m^{(0,\beta)}(x) \, dx = 0.$$

Therefore, one of the roots of $Q_m(k)$ must be given by $k = m - 1$. A similar procedure shows that for $k = m - 2, m - 3, \cdots, 1, 0$, the remaining roots of $Q_m(k)$ are determined. Therefore, $Q_m(k)$ is known up to a constant term, and it may be written in factored form as

$$Q_m(k) = A[k - (m - 1)][k - (m - 2)] \cdots k,$$

and A is a constant to be determined.

Substitution of $Q_m(k)$ as given here into (12) gives

$$(13) \qquad a_m h_m = \frac{A[k - (m - 1)]\,[k - (m - 2)] \cdots (k - 1)k}{(k + \beta + 1)\,(k + \beta + 2) \cdots (k + \beta + m + 1)}.$$

However, from (12) it follows that

$$\begin{aligned} A &= b_0 2^{k+\beta+1} + b_1 2^{k+\beta+2} + \cdots + b_m 2^{k+\beta+m+1} \\ &= 2^{k+\beta+1}[b_0 + 2b_1 + \cdots + 2^m b_m]. \end{aligned}$$

Since $P_m^{(0,\beta)}(1) = 1$ for $m = 0, 1, \cdots$, one has

$$P_m^{(0,\beta)}(1) = 1 = b_0 + 2b_1 + \cdots + 2^m b_m.$$

Hence, it follows that $A = 2^{k+\beta+1}$, and from (13) and some algebraic simplification

$$(14) \qquad a_m = 2^k(2m + \beta + 1)\frac{k(k - 1) \cdots [k - (m - 1)]}{(k + \beta + 1)(k + \beta + 2) \cdots (k + \beta + m + 1)}.$$

For $k = 0$ the right side of (14) is replaced by 1.

Substitution of (14) and (9) into (8) gives

$$(15) \qquad \begin{aligned} F[(\beta + 1 + k)\delta] &= \frac{2^{-(\beta+k+1)}}{\delta} \\ &\quad \cdot \int_{-1}^{1} (1 + x)^\beta \sum_{m=0}^{k} a_m\, P_m^{(0,\beta)}(x) \left[\sum_{n=0}^{\infty} C_n\, P_n^{(0,\beta)}(x)\right] dx \end{aligned}$$

for $k = 0, 1, \cdots$, where a_m is defined in (14). Integrating termwise in (15) gives only k nonzero terms because of the orthogonality property of the Jacobi polynomials. After the integration has been performed, algebraic simplification gives

$$(16) \qquad \begin{aligned} &\delta F[(\beta + 1 + k)\delta] \\ &\quad = \sum_{m=0}^{k} \frac{k(k - 1) \cdots [k - (m - 1)]}{(k + \beta + 1)\,(k + \beta + 2) \cdots (k + \beta + 1 + m)}\, C_m. \end{aligned}$$

Again this result is true for $k = 0, 1, \cdots$, and for $k = 0$ the right side of this expression is replaced by $c_0/(\beta + 1)$.

By successively allowing $k = 0, 1, \cdots$, one has the system of equations:

$$\delta F[(\beta+1)\delta] = \frac{C_0}{(\beta+1)},$$

$$\delta F[(\beta+2)\delta] = \frac{C_0}{(\beta+2)} + \frac{C_1}{(\beta+2)(\beta+3)},$$

$$\delta F[(\beta+3)\delta] = \frac{C_0}{(\beta+3)} + \frac{2C_1}{(\beta+3)(\beta+4)} \tag{17}$$

$$+ \frac{2C_2}{(\beta+3)(\beta+4)(\beta+5)},$$

$(\beta + 1 + 3)\delta$

$$\delta F[(\beta+4)\delta] = \frac{C_0}{(\beta+4)} + \frac{3C_1}{(\beta+4)(\beta+5)}$$

$$+ \frac{3\cdot 2C_2}{(\beta+4)(\beta+5)(\beta+6)} + \frac{3!C_3}{(\beta+4)\cdots(\beta+7)}.$$

The coefficient C_0 is determined by allowing $k = 0$ and knowledge of $F(s)$ at $s = (\beta+1)\delta$. For $k = 1$ the coefficient C_1 is determined from the value (calculated) of C_0 and $F(s)$ at $s = (\beta+2)\delta$. In a similar manner the remaining coefficients $C_2, C_3, \cdots$ can be determined.

If N coefficients are calculated, then $g(x)$ may be approximated by $g(x) \approx \sum_{n=0}^{N} C_n P_n^{(0,\beta)}(x)$. Since $x = 2\exp(-\delta t) - 1$, the Jacobi polynomials may be expressed as functions of t directly. From (4) it then follows that

$$f(t) \approx \sum_{n=0}^{N} C_n P_n^{(0,\beta)}[2\exp(-\delta t) - 1]. \tag{18}$$

Application of method. Theoretically, $f(t)$ can be determined for all values of t from knowledge of $F(s)$ at discrete points along the real s-axis. However, numerical errors limit the number of terms in (18) that can be accurately computed. Therefore, the accuracy of the approximation to $f(t)$ may be increased by selecting the position along the real s-axis at which $F(s)$ is evaluated. The points at which $F(s)$ is evaluated ($s = (\beta + 1 + k)\delta$ for $k = 0, 1, 2, \cdots$) are determined by β and δ. Thus, β and δ should be selected so that (in some sense) the "best" approximation possible is obtained.

It is well known that large s corresponds to small t and small s corresponds to large t, [3]. This fact is a guideline to follow and for asymptotic values of t the values of β and δ can be selected accordingly. Of more general interest, however, is the approximation of $f(t)$ for values of t which are not asymptotic. Therefore, for a given error norm, β and δ should be selected in order that the error is minimized.

Error bounds. Since the series in (7) converges uniformly (g continuous), it may be truncated after N terms to give an approximation valid for $x \in [-1, 1]$. Thus, there exists an $n_0 \geqq 0$ such that the terms in (7) are approaching zero. The rate of convergence of (7) may be used (or (18) in the t-space) as a criterion for selecting β and δ. First, however, some definitions are needed.

DEFINITION 1. Let g be continuous over $[-1, 1]$ and define $\epsilon_n(x)$ by

$$|\epsilon_n(x)| = \left| g(x) - \sum_{k=0}^{n} C_k P_k^{(0,\beta)}(x) \right|. \tag{19}$$

DEFINITION 2. The norm of the error in the approximation for $g(x)$ is defined by

$$\| \epsilon_n(x) \| = \int_{-1}^{1} |\epsilon_n(x)|^2 \, dx. \tag{20}$$

The theorem that follows gives an estimate of the error.

THEOREM 1. *Let g be continuous over $[-1, 1]$ and ϵ_n defined by* (19). *Assume that there exists a real number r, $0 < r < 1$, and there exists a positive integer p such that for $n \geqq p$,*

$$| C_{n+m} P_{n+m}^{(0,\beta)}(x) | \leqq r^m \, | C_n P_n^{(0,\beta)}(x) |,$$

for $m = 0, 1, \cdots$. Under these hypotheses it follows that for $n \geqq p$,

$$|\epsilon_n(x)| \leqq C_{n+1} P_{n+1}^{(0,\beta)}(x) | / (1 - r)^2. \tag{21}$$

Proof. Rewrite (19) in the form

$$|\epsilon_n(x)| = | C_{n+1} P_{n+1}^{(0,\beta)}(x) + C_{n+2} P_{n+2}^{(0,\beta)}(x) + \cdots |.$$

Application of the triangle inequality to this expression gives

$$|\epsilon_n(x)| \leqq | C_{n+1} P_{n+1}^{(0,\beta)}(x) | + | C_{n+2} P_{n+2}^{(0,\beta)}(x) | + \cdots.$$

Under the hypothesis of the theorem, use of the geometric series and some algebraic manipulation give the result in (21).

If $K = \max_{\beta,\delta} \{ | C_{p+1} P_{p+1}^{(0,\beta)}(x) | \}$, then it follows from (21) that $|\epsilon_n(x)| \leqq K/(1 - r)$. Hence, the following theorem gives a bound on the error norm $\| \epsilon_n(x) \|$.

THEOREM 2. *If $\epsilon_n(x)$ is defined by* (19) *and K is given as above, then for g continuous over* $[-1, 1]$,

$$\| \epsilon_n(x) \| \leqq 2K^2/(1 - r)^2.$$

Proof of Theorem 2 follows from the definition of $\| \epsilon_n(x) \|$ if $\epsilon_n(x)$ given in terms of K is substituted into (2).

A result similar to Theorem 2 holds in the t-space for any interval $(0, T)$.

THEOREM 3. *If* $e_n(t) = \epsilon_n(x)$ *and* x *and* t *are related by* (3), *then*

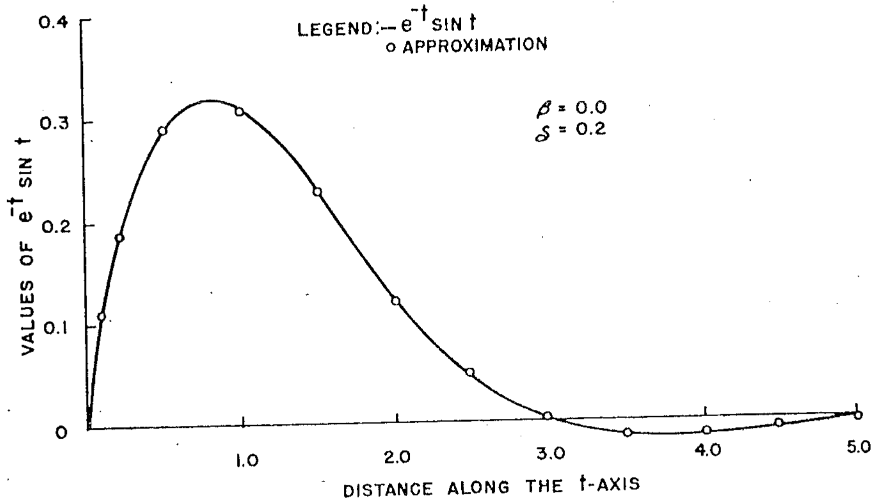

FIG. 1. *Approximations for* $f(t) = e^{-t} \sin t$

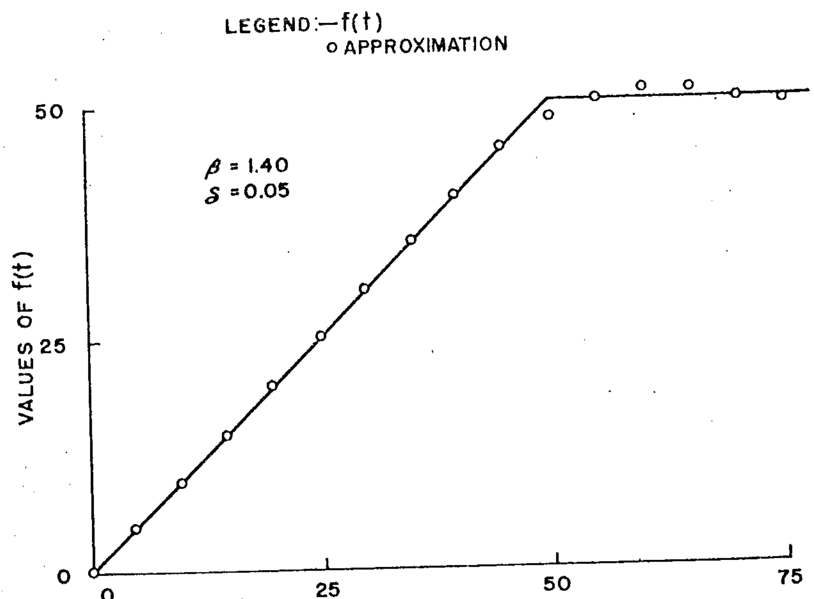

FIG. 2. *Approximations for* $f(t) = \begin{cases} t & \text{for} \quad 0 \leqq t \leqq 50, \\ 50 & \text{for} \quad 50 \leqq t \end{cases}$

$$\int_0^T | e_n(t)|^2 \, dt \leqq$$

Numerical examples. The exampl[…] indicate the results of this inversion […] they have poles at various positions […] used in the literature as examples […] cause the functions (in the t-space) d[…]

For the first example consider th[…] $= 1/[(s + 1)^2 + 1]$. The known i[…] results are shown in Fig. 1. For thi[…] terms were used in the approximatin[…]

The theory presented here require[…]

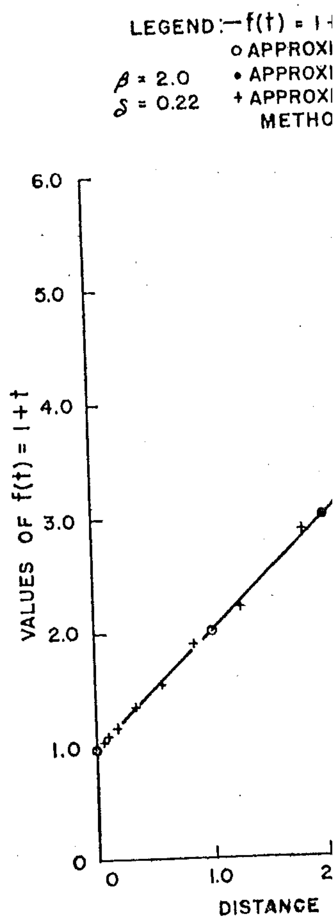

FIG. 3. *Approximations f*

$$\int_0^T |e_n(t)|^2\,dt \leqq \frac{1}{\delta}\, e^{\delta T} K^2/(1-r)^2.$$

Numerical examples. The examples given in the following paragraphs indicate the results of this inversion scheme. They were selected because they have poles at various positions in the complex plane, they have been used in the literature as examples of different inversion schemes, or because the functions (in the t-space) do not always have "gentle" slope.

For the first example consider the Laplace transform defined by $F(s) = 1/[(s+1)^2+1]$. The known inverse is $f(t) = \exp(-t)\sin t$. The results are shown in Fig. 1. For this calculation $\beta = 0.0$ and $\delta = 0.2$; 11 terms were used in the approximating function defined in (18).

The theory presented here requires that $f(0)$ and $f(\infty)$ be finite. Thus,

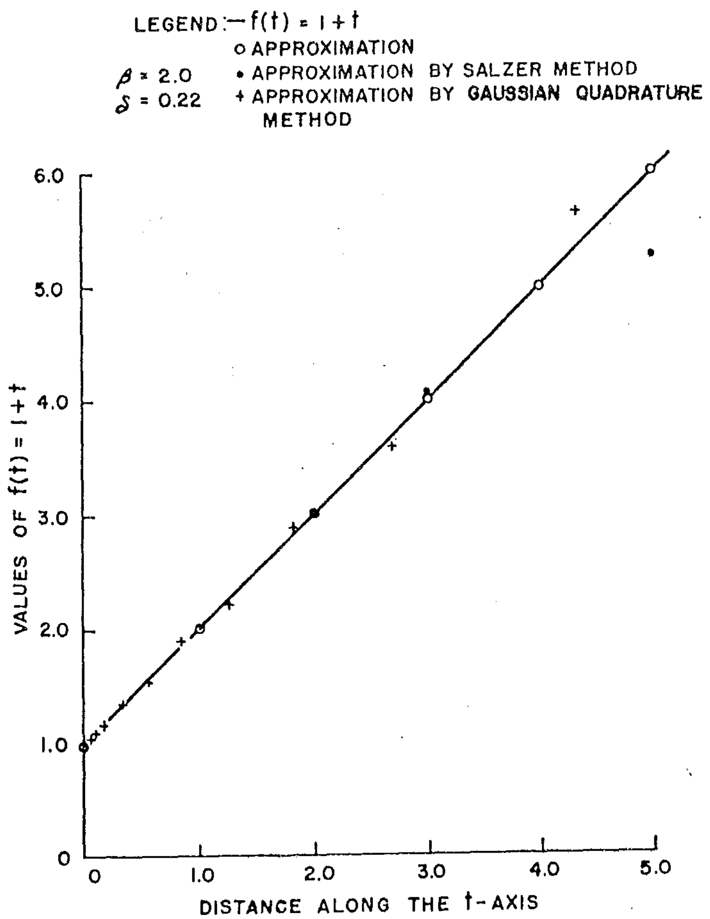

FIG. 3. *Approximations for* $f(t) = 1 + t$ *if ten terms are used*

for the Laplace transform $F(s) = 1/s^2$ which has an inverse $f(t) = t$, the theory is not applicable. However, the Laplace transform $F_1(s) = [1 - \exp(-st)]/s^2$ has an inverse $f_1(t) = t$ for $0 \leqq t \leqq T$ and $f_1(t) = T$ for $T \leqq t$. Hence for $T \to \infty$ and sT sufficiently large one has $\exp(-st) \ll 1$ and $F_1(s) \approx F(s)$. Fig. 2 shows the results obtained for $T = 50$. For these calculations $\beta = 1.40$, $\delta = 0.05$, and 11 terms were used in (18). For this approximation of $f(t)$ the range of values of t used is quite extensive with $0 \leqq t \leqq 75$.

As it was explained previously, for sT sufficiently large, $\exp(-st) \ll 1$. If this is true, then on the register of a computer $F_1(s) = F(s)$. That is, for sufficiently large s, the technique can be applied to $F(s) = 1/s^2$. Fig. 3 shows the results obtained for the approximation to $f(t) = t$, where $0 \leqq t \leqq 5$ and $\beta = 2.0$ and $\delta = 0.22$. Two other known methods were also used to numerically invert $F(s) = 1/s^2$. One of these methods is due to Salzer [7], [8] and the other method uses a Gaussian type quadrature [1], [2], [4]. In each of the approximation schemes a 10-point quadrature [10 terms in (18)] was used. That is, $F(s)$ was evaluated at 10 points along the real s-axis. Tables used for these comparisons were obtained from [1], [7].

Fig. 4 shows the results obtained for $f(t) = J_0(t)$. The approximations again use 10 terms. $F(s)$ was evaluated at points determined by $\beta = 3.0$ and $\delta = 0.5$. Values of t are for $0 \leqq t \leqq 5$. For a specific value of t a different choice of β and δ gives better results. For this example it was found that for $J_0(2)$, the values $\beta = 4.0$ and $\delta = 0.6$ give the approximation $J_0(2) \approx 0.223896$, while $J_0(2) = 0.223891$ (rounded to six decimal digits).

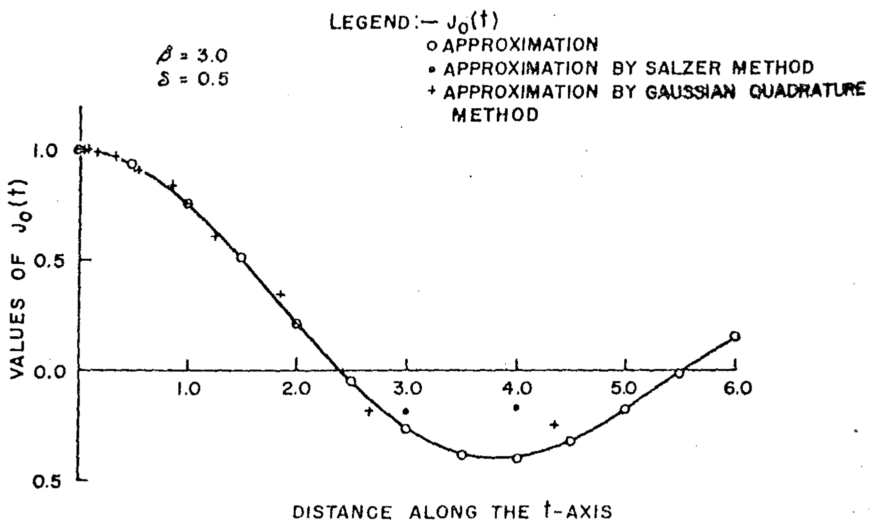

FIG. 4. *Approximations for $J_0(t)$ if ten terms are used*

The next example is for the Laplace t
inverse is given by

$$f(t) = \frac{\exp}{4(}$$

This example is given by Bellman, et a
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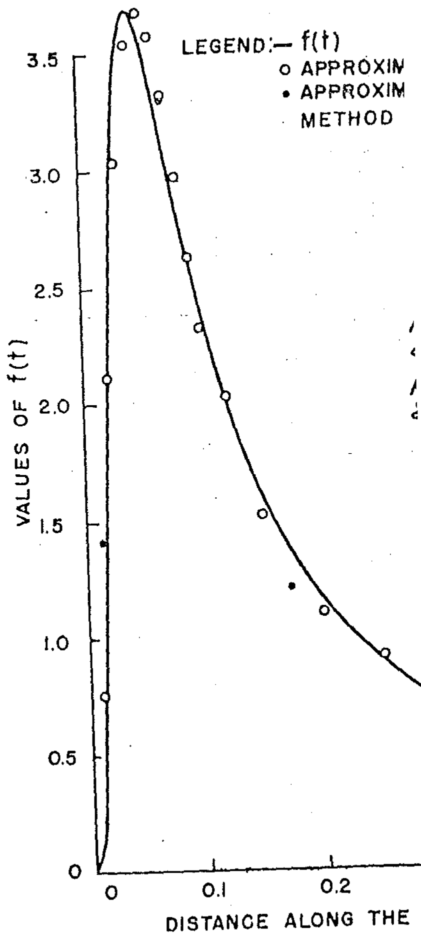

FIG. 5 *Approximations*

The next example is for the Laplace transform $F(s) = \exp(-\frac{1}{2}\sqrt{s})$. The inverse is given by

$$f(t) = \frac{\exp(-t/16)}{4(\pi t^3)^{1/2}}.$$

This example is given by Bellman, et al., [2] and illustrates the difficulty involved in numerically inverting a Laplace transform which has an inverse with a "steep" slope. Ten terms in (18) and a 10-point quadrature were used for these calculations. One of the advantages of the method described here is illustrated in this example. This is the fact that $f(t)$ may be approxi-

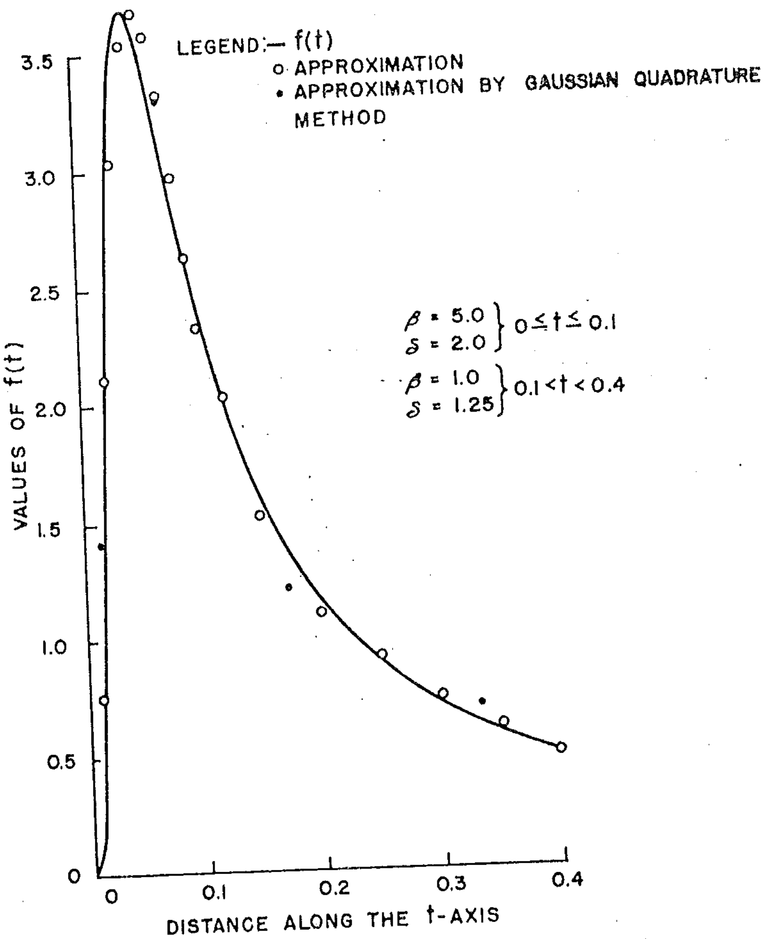

FIG. 5 *Approximations for* $f(t) = \dfrac{\exp(-t/16)}{4(\pi t^3)^{1/2}}$

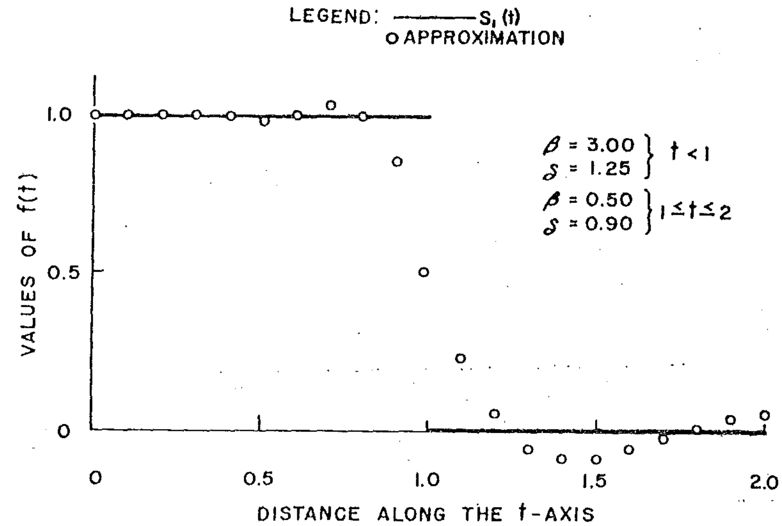

FIG. 6. *Approximations for* $f(t) = S_0(t)$

mated at values of t which lie sufficiently close so that the outline of $f(t)$ is well described, as shown in Fig. 5.

The previous examples have been for continuous functions in the t-space. The infinite series representation for these functions converges uniformly and the termwise integration in (15) is valid. Consider the step function given by $S_1(t)$ for values of t in $(0, 2)$. Although it has not been shown that the termwise integration in (15) may be performed without altering the results, a "rough" outline of $f(t)$ may still be obtained in this particular example. Fig. 6 shows these results.

Numerical errors. Numerical round-off and cancellation errors limit the number of coefficients c_n that can be accurately calculated from the system of equations in (17). By the use of multiple precision arithmetic, the number of coefficients that may be accurately computed is increased. The exact number of coefficients which can be accurately computed depends on a particular problem. Experience has shown that for these examples and for ones similar, about 12 to 14 coefficients may be accurately calculated using single precision arithmetic on a Control Data 1604 computer.

The Jacobi polynomials were calculated using the recurrence relation found in [9, p. 71].

Conclusions. The method for numerically inverting Laplace transforms that has been described here is applicable to many problems of practical interest. Round-off and cancellation errors must be considered when calculating the coefficients that appear in the series approximation for $f(t)$. For

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a small number of calculations $f(t)$ may be approximated over a wide range of values. A general guide for the user of this method is to select β and δ such that $-0.5 \leqq \beta \leqq 5.0$ and $0.05 \leqq \delta \leqq 2.0$. For t such that $t > 0.1$, a more realistic value of β is $\beta \leqq 2.0$. The required computer time is only a fraction of a second for computation of 15 Jacobi polynomials and 15 coefficients on a Control Data 1604 computer.

Acknowledgment. The support of the facilities of the Computation Center at the University of Texas is gratefully acknowledged.

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Numerical Evaluation of Cumulative Probability Distribution Functions Directly from Characteristic Functions

Abstract—A method for direct numerical evaluation of the cumulative probability distribution function from the characteristic function in terms of a single integral is presented. No moment evaluation or series expansions are required. Intermediate evaluation of the probability density function is circumvented. The method takes on a special form when the random variables are discrete.

Introduction

It often happens in engineering calculations involving random variables that it is difficult to obtain direct values of the cumulative probability function but relatively easy to obtain values or a closed-form expression for either the moment-generating function or the characteristic function. In a recent letter, Helstrom[1] presented a technique for calculating cumulative probabilities from a moment-generating function. We wish to present an alternative numerical technique for calculating the cumulative probability from the characteristic function, defined only on the real axis.

General Distributions

The general case follows directly from equation (4.14) of Kendall and Stuart:[2]

$$P(X) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty}\frac{d\xi}{\xi}\left[\mathrm{Re}\,\{f(\xi)\}\sin(\xi X) - \mathrm{Im}\,\{f(\xi)\}\cos(\xi X)\right], \quad (1)$$

where $P(X)$ is the cumulative probability distribution function (CDF), and $f(\xi)$ is the characteristic function (CF) of a random variable x. At a point of discontinuity of the CDF, (1) takes on the mid-value.[3]

The integral in (1) is confined to the real axis. Since some CF's exist only for real ξ (for example, $\exp(-|\xi|)$), (1) is a useful and general form for computational purposes. The CF does not have to be analytic at the origin.

Discrete Distributions

The expression (1) requires an infinite integral for each value of X. Here we eliminate this requirement for a special class of random variables. Specifically, we consider discrete random variables that can take on only values that are multiples of some fundamental increment Δ. That is, the probability density function (PDF) of interest takes the form

$$p(x) = \sum_k c_k \delta(x - k\Delta). \quad (2)$$

(A sum without limits is over the integers from $-\infty$ to $+\infty$.) Then the CF is

$$f(\xi) = \sum_k c_k \exp(ik\Delta\xi), \quad (3)$$

which is periodic with period $2\pi/\Delta$. Therefore, the coefficients $\{c_k\}$ can be determined from the CF $f(\xi)$ by

$$c_k = \frac{\Delta}{2\pi}\int_{2\pi/\Delta} d\xi \exp(-ik\Delta\xi) f(\xi), \quad (4)$$

where the integral is over any interval of length $2\pi/\Delta$.

Equation (4) gives the area of any impulse in the PDF $p(x)$ in terms of a finite integral of the CF $f(\xi)$. Since we are interested in the CDF $P(X)$, a sum over $\{c_k\}$ is required. At this point, we restrict consideration to nonnegative discrete random variables. (Extensions to general discrete random variables have been developed by Nuttall.[4]) At integer value M, the CDF becomes

$$P(M) = \sum_{k=0}^{M} c_k = \frac{\Delta}{2\pi}\int_{2\pi/\Delta} d\xi f(\xi) \sum_{k=0}^{M}\exp(-ik\Delta\xi)$$
$$= \frac{\Delta}{\pi}\int_0^{\pi/\Delta} d\xi \frac{\sin[(M+1)\Delta\xi/2]}{\sin[\Delta\xi/2]}\mathrm{Re}\,\{f(\xi)\exp(-iM\Delta\xi/2)\}, \quad M \geq 0, \quad (5)$$

where the interval $(-\pi/\Delta, \pi/\Delta)$ has been selected for integration, and we have used the property $f(-\xi) = f^*(\xi)$. (The ratio of sines is interpreted as $M+1$ at the origin.) Equation (5) is a single finite integral from which the CDF $P(M)$ can be evaluated at any M directly from the CF $f(\xi)$.

TABLE I

Numerical Computation of Exponential Distribution

X	$P(X)$	Finite Sum via (1)	Increment in ξ
−10	0	0.000 01	0.1
−2	0	−0.000 07	0.5
−1	0	0.000 08	0.5
0	0	0.005 32	0.5
0.2	0.181 27	0.180 96	0.5
1	0.632 12	0.632 20	0.5
2	0.864 66	0.864 70	0.5
10	0.999 954 6	0.999 963 7	0.1

TABLE II

Numerical Computation of Poisson Distribution

M	$P(M)$	Finite Sum via (5)
0	0.000 000 305 9	0.000 000 305 9
1	0.000 004 894 4	0.000 004 894 5
6	0.007 631 899 6	0.007 631 899 8
14	0.465 653 708 9	0.465 653 708 9
16	0.664 123 200 5	0.664 123 200 4
20	0.917 029 089 9	0.917 029 089 5
29	0.999 581 550 2	0.999 581 550 0
30	0.999 802 686 7	0.999 802 686 5
40	0.999 999 976 5	0.999 999 976 4

Examples

We shall consider two examples recently examined by Helstrom[1] for purposes of comparison.

Example 1: Exponential Distribution

$$p(x) = \begin{Bmatrix} \exp(-x), & x \geq 0 \\ 0, & x < 0 \end{Bmatrix}, \quad (6)$$

$$f(\xi) = (1 - i\xi)^{-1}. \quad (7)$$

The integral of (1) was sampled in ξ at values indicated in column four of Table I and approximated by the trapezoidal rule for integration. The limit of integration in (1) was taken to be the value above 60 where the finite sum deviated most from the exact answer. Thus the finite sum in column three of Table I is the worst approximation to the exact answer in column two.

Example 2: Poisson Distribution

$$p(x) = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta(x-k), \quad (8)$$

$$f(\xi) = \exp[\lambda\{\exp(i\xi) - 1\}]. \quad (9)$$

The integral of (5) was divided into 25 equal intervals and approximated by the trapezoidal rule for integration. Columns two and three of Table II show that the error in the approximation occurs in the tenth place (and may be due to computer inaccuracies rather than sampling errors). Also, the accuracy holds on the tails of the CDF as well as near the mean.

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Conclusions

The numerical technique suggested for obtaining CDF's directly from CF's has considerable merit. It requires no moment evaluations or series expansions (like the techniques of Edgeworth or Laguerre) for the distributions. It does not depend upon evaluation of derivatives of CF's but only upon the values of the CF itself. (Evaluation of high-order derivatives can be extremely tedious and time-consuming even if an analytic form for the CF is available.) The accuracy of the suggested technique can be estimated and controlled by decreasing the increment in the integral evaluations or lengthening the interval of integration, or both; the change in the approximation is a measure of the error at that point. The method does not require an inordinate number of samples of the CF, at least for the examples considered, and the additional functions requiring evaluation are sines and cosines. Intermediate evaluation of the PDF is entirely circumvented.

Albert H. Nuttall
Navy Underwater Sound Lab.
New London, Conn. 06320

Saturation of Zn–O Complexes in GaP Diodes

Abstract—The red electroluminescence in gallium phosphide at the maximum quantum efficiency is a constant, independent of injection efficiency, for a series of liquid-phase epitaxially grown diodes which have common Zn- and O-doped p-type substrates and variable Te-doped n-type layers. This behavior and the subsequent decrease in quantum efficiency with increasing diode current are both explained in terms of the saturation of Zn–O complexes by captured electrons in the p region.

The room-temperature red luminescence in p-type GaP doped with Zn and O has been identified as radiative recombination of excitons bound to nearest neighbor Zn–O complexes [1], [2]. It has been observed that the electroluminescent (EL) intensity in p-n junction GaP diodes varies linearly with electron injection level at low levels and sublinearly at high levels, causing the diode quantum efficiency to pass through a maximum and to decrease [3], [4]. In a series of liquid-phase epitaxially grown (LPE) diodes having common Zn- and O-doped p-type substrates and variable Te-doped n-type layers, we find that the total red EL intensity at the maximum quantum efficiency is a constant, independent of injection efficiency. This behavior is due to the saturation of the bound exciton population, limited to the same value in each diode by the fixed Zn–O complex concentration [5]. However, the forward-bias dependence of the EL intensity differs significantly from that previously reported [3], [4]. Below saturation the intensity varies as exp (qV/kT) as before; above saturation the intensity varies as qV/kT, which is a marked departure from the previously reported dependence of exp $(qV/2kT)$. The previous behavior had been explained in terms of either a single saturable radiative recombination route for injected electrons in the p region [6] or in terms of space-charge recombination [4]. The new behavior reported in this letter is explained in terms of recombination in the p region, using both a saturable radiative route via Zn–O complexes and a faster nonsaturable route which dominates the minority carrier lifetime (detailed analysis is given in [7]). Within this framework, the Zn–O complex concentration and the capture cross section for electrons can be calculated from the bias-dependent EL intensity and other experimental data [7], [8].

The p-n junctions were prepared by growing Te-doped n layers onto solution-grown Zn- and O-doped p-type substrates via an LPE process. The substrate material was doped with 0.07 mole percent Zn and 0.02 mole percent Ga_2O_3, reported to be an optimal doping for the red luminescence [9]. Five groups of p-n junctions were grown with Te concentrations varying from 0.0035 to 0.079 mole percent in the melt. Mesa diodes were fabricated from as-grown p-n layers, and from p-n layers which had been annealed in forming gas (15 percent $H_2 + 85$ percent N_2) at 600°C for 6 hours.

The external quantum efficiency η for each diode was measured in an

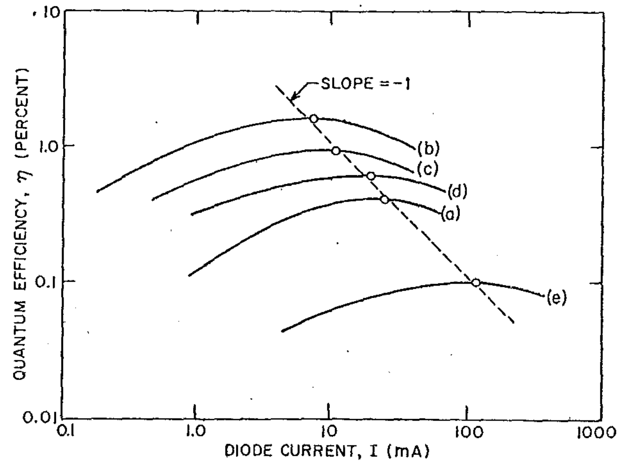

Fig. 1. Current dependence of the total quantum efficiency for a representative diode tak from each of five groups of annealed diodes. Each group was prepared with different concentrations in the melt: 0.0035 mole, percent, 0.009 mole percent, 0.018 mole perce 0.028 mole percent, and 0.079 mole percent. Junction areas are 7×10^{-4} cm^2. At hi current levels (2–500 mA), measurements were made on a pulsed biasis to elimin heating (i.e., 10 μs pulse at 1 percent duty cycle).

integrating sphere over several decades of current. As shown in Fig. 1, maximum is observed for a representative annealed diode from each grou The open circles in Fig. 1 indicate the maximum efficiency η_0 and th current I_0 at the maximum efficiency. Although η_0 and I_0 vary widel with each group of annealed diodes, an inverse relationship exists, i.e $\eta_0 \propto I_0^{-1}$. Thus, at I_0 the red external EL intensity L_0 is the same for a annealed diodes, $L_0 = \eta_0 I_0/q$, independent of tellurium concentration an independent of injection efficiency (q is the electronic charge). This be havior is observed for both as-grown and annealed diodes, with a 2 percent increase in L_0 observed in the annealed diodes [5]. Electrolumi nescent spectra taken on the two groups of diodes show that the componen at 1.36 eV, which has been attributed to infrared O-donor Zn-accepto distant pair recombination [10], [11], decreases from 5.3 percent of the 1.8 eV red peak for as-grown diodes to 3.6 percent of the red peak fo annealed diodes, indicating a formation of Zn–O complexes during annealing [11].

The fact that the red external electroluminescence L_0 is constant at I_0, independent of tellurium concentration and independent of the p-n junction injection efficiency, indicates a saturation process characteristic of the p region[1] [3]. Since the p regions of all the diodes have the same Zn concentrations and the same O concentrations and have gone through the same temperature cycling during the liquid-phase epitaxial process, we expect them to have nearly equal concentrations of Zn–O complexes. Assuming that the complexes can be saturated with trapped electrons at high electron injection levels, the bound exciton concentration should be limited to equal levels in all the diodes, independent of the electron injection efficiency. Thus the red EL intensity should be limited to approximately the same level in each diode at the onset of complex saturation. This is the point at which the red EL intensity becomes sublinear with injection level and the quantum efficiency passes through a maximum and begins to decrease with diode current.

It is interesting to note that for Te concentrations above 0.009 mole percent, the maximum EL efficiency decreases with an increase in Te concentration. This result seems to indicate that the electron injection into the p region becomes less efficient with an increase in Te concentration, which is not at all what one would expect on the basis of a simple abrupt junction calculation.

While the saturation behavior described above is consistent with the observations and interpretations given in [3], the bias dependence of the EL intensity in the saturation regime displays a striking difference (see Fig. 2). In saturation the intensity varies as qV/kT, in contrast to the

[1] We are neglecting changes in the bulk absorption coefficient with Te in our diodes since the Te-doped n-type layer constitutes approximately 10 percent of the total diode volume.

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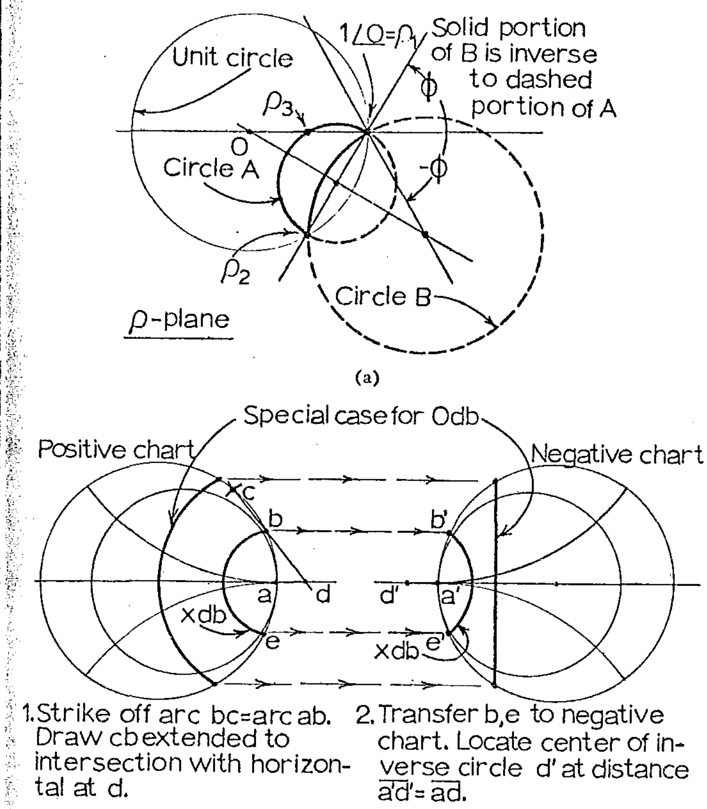

Fig. 2 (a). Construction of inverse circle. (b) Construction of constant gain (dB) loci from opposite chart.

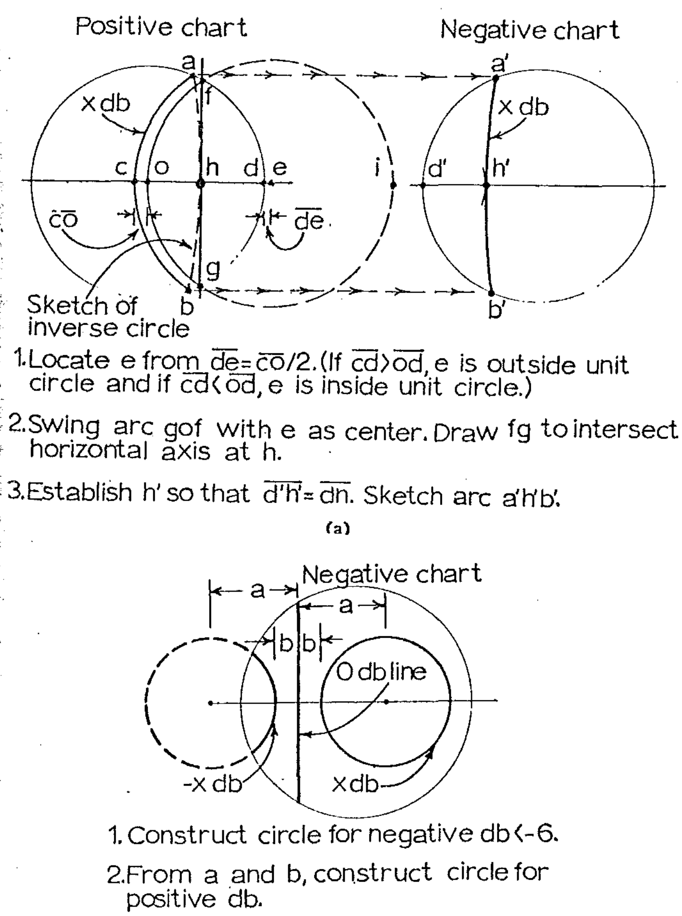

Fig. 3 (a). Construction of inverse circle for small dB. (b) Construction of negative chart circle for dB greater than six.

which, as shown in Fig. 2(a), is a straight line passing through $1/\underline{0}$ and making an angle ϕ with the horizontal axis.

Since A and B intersect the unit circle at the same points, the center of B can be found directly by drawing a straight line through the origin and the center of A. The intersection with the plot of (5) establishes the center of B. If, in addition, A passes through the origin, the circle B becomes a chord passing through the intersection points.

Because of the mapping that establishes the negative Smith chart used by McNaughton and West [1], a locus on one chart for a given dB value is an inverse circle (within a physical rotation of 180°) to the locus for the given dB on the other chart.

Constant gain circles on the positive chart present a special case of Fig. 2(a) for circles whose centers are known to lie on the circumference of a unit circle. This leads to the particularly simple graphical means of establishing the line that intersects the horizontal axis at d, as explained in the instructions of Fig. 2(b). Transfer of the points, d, e, and b to the negative chart establishes the constant gain circle on the negative chart, as illustrated in Fig. 2(b). The negative chart inverse "circle" (a straight line) for the zero dB case is shown for reference.

Except for the zero dB locus itself, the negative chart constant gain circles for small dB cannot be constructed conveniently because the centers lie far off the chart. However, because any circle passing through the center and circumference of a unit circle has a chord as its inverse, it is possible to determine the point h [see Fig. 3(a)] where the desired inverse constant gain circle intersects the horizontal axis. The construction shown on Fig. 3(a) establishes the three points a', h', and b' on the negative chart through which sketched curves, or even straight lines, can be drawn, depending on the degree of accuracy desired for the small dB locus.

Finally, constant gain curves for dB values in excess of six do not appear at all on the positive chart and appear on the negative chart as circles within the chart. From the McNaughton and West [1] equation for constant gain circles on the negative chart, it can be seen that circles for the same dB magnitude have the same radii and are symmetrically located with respect to the zero dB chord. Thus, for $x>6$, the x dB circle on the negative chart is found from the $-x$ dB circle on the negative chart which, in turn, is found from the $-x$ dB circle on the positive chart. Construction is shown in Fig. 3(b).

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Laguerre? Bessel (Not Computer)

Notes on Formal Expansion Techniques Involving Laplace Transforms

O. R. AINSWORTH and C. K. LIU

The Newmann expansion of an analytic function in the series

$$f(z) = f(0)J_0(z) + \sum_{n=1}^{\infty} A_n J_n(z)$$

where

$$A_n = \frac{1}{\pi i} \int_{|z|=r} 0_n(t) f(t)\, dt$$

is, of course, well known. However, the computation of the coeffi-

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The authors are with the University of Alabama, University, Ala.

except in trivial cases, is rather difficult and prevents one ely using the expansion. This expansion converges so rapidly s really quite a desirable one.

now offer a much easier way of computing the A_n for a large functions $f(t)$.

the Laplace transform of $f(t)$ be given by $F(s)$. Note that

$$L\left\{\frac{a^k k}{t} J_k(at)\right\} = [\sqrt{s^2 + a^2} - s]^k.$$

:

$$P = \sqrt{s^2 + a^2} - s$$

ıfter trivial arithmetic

$$s = \frac{a^2 - p^2}{2p}.$$

have

$$L\{f(t)\} = F(s) = F\left(\frac{a^2 - p^2}{2p}\right) = \sum_k b_k p^k.$$

nition of p was such that

$$L\left\{\frac{a^k k}{t} J_k(at)\right\} = p^k.$$

efore invert both sides of $L\{f(t)\}$ and obtain

$$f(t) = \sum_k b_k \frac{a^k k}{t} J_k(at).$$

Hence,

$$tf(t) = \sum_k b_k a^k k J_k(at).$$

Of course, we could have taken the transform of $f(t)/t$ by using a series technique, if necessary, and obtained the Newmann expansion for $f(t)$ itself. Also, from Table I we observe the natural extensions to expansions in $I_k(at)$, $e^{-1/2at}J_k(at)$, $t^{k-1/2}J_{k-1/2}(at)$, etc. Table I is by no means complete since the elementary techniques in Laplace transform theory give rise to a large number of trivial variations. This method apparently was known to Cailler in 1905.[1]

An important variation—namely, 4) and 5) of Table I—will permit us to expand functions into the series

$$f(t) = \sum_{n-1}^{\infty} b_n t^{n-1/2} J_{n-1/2}(at).$$

Of course, this has ceased to be a Newmann expansion, but it is of interest in itself. It is an easy extension of the Cailler method, but apparently has been overlooked.

Another easy variation would be to use 6) in Table I in the expansion of the Laguerre polynomial. Here

$$L\{L_n(t)\} = \frac{1}{s}\left\{\frac{s-1}{s}\right\}^n$$

[1] See G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, England: Cambridge University Press, 1944, p. 536.

TABLE I

1) $L\left\{\frac{ka^k}{t} I_k(at)\right\}$	$= [s - \sqrt{s^2 - a^2}]^k$	$= p^k$	where	$s = \frac{p^2 + a^2}{2p}$
2) $L\left\{\frac{k}{t} e^{-(1/2)at} I_k(\frac{1}{2}at)\right\}$	$= \left[\frac{a}{\sqrt{s+a} + \sqrt{s}}\right]^k$	$= (ap)^k$	where	$s = \left(\frac{ap^2 - 1}{2p}\right)^2$
3) $L\left\{\frac{ka^k}{t} J_k(at)\right\}$	$= [\sqrt{s^2 + a^2} - s]^k$	$= p^k$	where	$s = \frac{a^2 - p^2}{2p}$
4) $L\left\{\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2} J_{k-1/2}(at)\right\}$	$= \left(\frac{1}{s^2 + a^2}\right)^k$	$= p^{2k}$	where	$s = \frac{\sqrt{1 - a^2p^2}}{p}$
5) $L\left\{\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)\right\}$	$= \left(\frac{1}{s^2 - a^2}\right)^k$	$= p^{2k}$	where	$s = \frac{\sqrt{1 + a^2p^2}}{p}$
6) $L\{L_k(t)\}$	$= \frac{1}{s}\left(\frac{s-1}{s}\right)^k$	$= \frac{1}{s} p^k$	where	$s = \frac{1}{1-p}$

TABLE II

1) $te^{-at} = \frac{2}{a} \sum_{n=0}^{\infty} (-1)^n (n+1)^2 I_{n+1}(t)$

2) $\operatorname{Erfc} \frac{x}{2\sqrt{ht}} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{n+l} x^n a^{(1/2)n-1}}{h^{(1/2)n} n! l! 2^{(1/2)n-1}} \frac{2l - \frac{1}{2}n + 1}{t} J_{2l-(1/2)n+1}(at)$

3) $Si(ht) = \frac{1}{t} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^n (2h)^{2n+2}}{l! a^{2n+2}} \frac{\Gamma(2n + 2 + l + 1)}{\Gamma(2n+2)} \frac{2n + 2 + 2l}{2n+1} J_{2n+2+2l}(at)$

4) $\sin at = \frac{a\sqrt{\pi}}{\Gamma(1)} \left(\frac{t}{2a}\right)^{1/2} J_{1/2}(at)$

5) $\cos at = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{a^n}{n!(n - \frac{1}{2})} \left(\frac{1}{2} t\right)^n J_n(at)$

6) $t^{l+1} = \frac{2^{l+1}}{a} \sum_{n=0}^{\infty} \frac{\Gamma(l + n + 1)}{n!} (2n + 2l + 1) J_{2n+2l+1}(at)$

7) $t^n = \sum_{k=0}^{n} \frac{(n!)^2 (-1)^k}{k!(n-k)!} L_k(t)$

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that set

$$p = \frac{s-1}{s}, \qquad s = \frac{1}{1-p}$$

and then if

$$L\{\ \} = F(s) = \frac{1}{s}\left[\frac{1}{1-p} F\left(\frac{1}{1-p}\right)\right] = \frac{1}{s}\sum b_n p^n$$

we immediately have

$$f(t) = \sum b_n L_n(t)$$

which avoids the evaluation of the integrals

$$\int_0^\infty e^{-t} f(t) L_n(t) dt.$$

There exist several trivial variations on this theme derived from combinations of the Laplace transform in $L\{e^t f(t)\}$, $L\{t^\alpha f(t)\}$, etc.

This last procedure is apparently known to several investigators, and it is difficult to assign priority to any one of them. One really uses the idea of this method even as a sophomore when faced with a transform $F(s)$ which is not to be found in the tables. One merely expands

$$F(s) = \sum_n b_n \frac{1}{s^n}$$

and then inverts, getting

$$f(t) = \sum_n \frac{b_n}{\Gamma(n+1)} t^{n+1}.$$

Unfortunately, power series very often converge so slowly that many terms are required. The behavior of $J_n(z)$ for large n is such that there is rarely any need for more than the first few terms.

Table II lists only a few of the expansions we have obtained.

Polynomial Root Determination by an Equivalent Transfer Function Simulation on an Iterative Analog Computer

E. F. RICHARDS

Abstract—**The methods for determining the roots of polynomials by analog computer techniques are varied, but stability analysis is usually of chief concern. The computer approach used in this paper always produces stable operation, and introduces an equivalent transfer function representation of the polynomial, a linear transformation, and a generalized iterative analog computer program. The accuracy obtainable is not comparable to that of digital programs; however, the procedure is basic to engineering analysis and should not be overlooked from the academic viewpoint.**

Introduction

The usual process of obtaining the roots of polynomials with the analog computer consists of a trial-and-error technique using open-loop programming methods; this procedure can be unstable and consequently may make root determination very difficult. Bush and others [1], [2] have presented methods by which closed-loop techniques can be used to obtain solutions for sets of linear and nonlinear algebraic equations. Analog computers, by design, are dynamic tools and do not lend themselves readily to steady-state problems. The method to be presented here uses a closed-loop form of solution. An analogy is made between the polynomial whose roots are desired and a transfer function whose characteristic roots are identically the roots of the original polynomial.

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The author is at the University of Missouri, Rolla, Mo.

Theory

The process consists of the following four steps:

1) Formulation of the analogous transfer function and the direct programming technique to obtain a convenient state diagram.
2) An S-plane transformation and a corresponding modification to the state diagram.
3) Writing the iterative analog computer program directly from the state diagram and determining the real parts of the roots in order: $\alpha_1, \alpha_2, \cdots, \alpha_n$ etc.
4) Determining the next root of the transfer function after first either directly dividing out the roots as they are found in the characteristic equation or adding corresponding zeros to the analog computer program.

The programming technique as suggested in steps 1) and 2) leads directly to the analog computer program for obtaining the solution to the characteristic equation (with the possible exception of sign changes which are inherent in electronic operational amplifiers). The S-plane axis transformation in step 2) is quite direct and is a well-known analytical procedure. It has been used successfully by trial-and-error methods employing Routh's criterion to determine the roots of equations. However, the technique used here is believed to be a new analog computer approach to polynomial root determination.

Procedure

The four steps in the procedure will now be considered in greater detail.

1) *Formulation of Transfer Function and State Diagram*

Consider the general equation whose roots are desired:

$$X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1X_1 + a_0 = 0, \tag{1}$$

where the coefficients are constants.

By assumption, this can be written as a transfer function of the general form:

$$G(s) = \frac{1}{S^n + a_{n-1}S^{n-1} + a_{n-2}S^{n-2} + \cdots + a_1S + a_0}. \tag{2}$$

By direct programming, this can be reduced to a convenient state diagram, which is the analog computer program directly; the output of the integrators constitutes here one set of state variables for the particular problem.

The general form of the transfer function can be rewritten in a convenient form for direct programming:

$$G(s) = \frac{1}{[[[(S + a_{n-1})S + a_{n-2}]S + a_{n-3}]S + \cdots + a_1]S + a_0}. \tag{3}$$

The corresponding program is shown in Fig. 1.

2) *S-Plane Transformation and Modification of the Direct Program*

By applying the transformation $S = \bar{S} - \alpha$ to (3), one can shift the axis of the reals by a factor α. The procedure then is to obtain a convenient way to increment α in the analog program so that the roots desired can be shifted to either the right or the left half of the S plane. To make the process general, a convenient modification of the program of step 1) must be obtained.

When the transformation $S = \bar{S} - \alpha$ is applied to (3),

$$G(\bar{S}) = \frac{1}{[[[(\bar{S} - \alpha + a_{n-1})(\bar{S} - \alpha) + a_{n-2}](\bar{S} - \alpha) + a_{n-3}](\bar{S} - \alpha) + \cdots + a_1](\bar{S} - \alpha) + a_0}. \tag{4}$$

The modified program appears in Fig. 2.

TABLE 1

R_j

0	G	I	S	E	$\odot_a$	$\odot_b$	)	]	,	;	E_{rp}	E_{rb}	E_t	RH
	N													
	N*				N_2									
	N*				N_2									
	N*													
N_1	G				$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				
N_1	N				$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				
	N*				$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				
	I	I	V	F	$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				Σ
					$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				Σ
					$\odot_u$	$\odot_u$	E_{rp}	E_{rb}	E_c	E_t				
											E_{rp}	E_{rb}	E_t	
											E_{rp}	E_{rb}	E_t	
											E			
												S		
											E_{rp}	E_{rb}		
(													RH	

The characters introduced by the substitution process have the following meanings:

- G — an integer
- N — a number containing a decimal point
- N_1 — an incomplete number, ending in $_{10}$
- N_2 — an incomplete number, ending in $_{10}\pm$
- N^* — a number ending with an exponent of 10
- I — an identifier; a letter followed by letters or digits
- V — a subscripted variable
- E — a parenthesized expression
- S — a bracketed subscript
- $\odot_u$ — a unary arithmetic operator
- $\odot_b$ — a binary operator
- $\odot_a$ — an ambiguous operator (+ or −), unary or binary according to context
- E_c — an expression followed by a comma
- E_{rp} — an expression followed by a right parenthesis
- E_{rb} — an expression (or list of expressions separated by commas) followed by a right bracket
- E_t — an expression followed by a semicolon
- RH — the replacement operator := followed by E_t
- Σ — an identifier or subscripted variable followed by RH; a well-formed formula
- F — a function

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Numerical Inversion of Laplace Transforms*

GAUS QUAD.

LOUIS A. SCHMITTROTH, *Phillips Petroleum Co., Idaho Falls, Idaho*

roduction

note describes a method for computing the inverse aplace transform $F(s)$, when it is known that all rities of $F(s)$ lie in the left half-plane, $\text{Im}(s) < 0$. ethod has been programmed for the IBM 650 and tory results obtained. Some limitations and pos-tensions will be indicated below.

impetus for the development of the program came problem in the design of a reactor control system. ntrol system under consideration uses two control one of which has two time delays, so that the result-nsfer function is of a complicated type involving ntials in a nontrivial manner. It seemed computa-y prohibitive to try the traditional approach of and residues, so the present direct method was de-d.

ork d. . under contract to the U. S. Atomic Energy Com-n.

2. The Complex Inversion Integral

If a given function $F(s)$ fails to fall into a table of Laplace transforms, the usual procedure is to try to invert it by use of the complex inversion integral:

$$I(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds. \qquad (1)$$

Here c is any real constant such that all singularities of $F(s)$ are in $\text{Im}(s) < c$.

It is assumed that $F(s)$ has an inverse $f(t)$ (continuous and of exponential order) and that the inversion integral represents $f(t)$ in the sense that (see Churchill [1], Ch. 6):

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds = \begin{cases} 0, & t < 0, \\ \frac{1}{2}f(0+), & t = 0, \\ f(t), & t > 0. \end{cases} \qquad (2)$$

1960 vol 3

If $F(s)$ is of a simple nature, e.g., a rational function, one can find the poles and residues and hence evaluate $f(t)$. However, it often happens that the poles and residues cannot be found without a prohibitive amount of computation and a direct numerical method must be used. Also, the function $F(s)$ may be known only from empirical data, in which case direct numerical inversion is the only practical way.

3. Resolution into Trigonometric Integrals

As mentioned above, all singularities of $F(s)$ are assumed to lie in $\mathrm{Im}(s) < 0$, hence we may take $c = 0$. Furthermore we need $f(t)$ only for $t > 0$, so that our formula is:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t} \, d\omega, \quad t > 0. \tag{3}$$

Since $F(s) = \int_0^{\infty} f(t) e^{-st} dt$ is real for $s > 0$ in practical problems, we may assume that $F(\bar{s}) = \overline{F(s)}$, where the bar denotes complex conjugation. We will use the definitions $\varphi(\omega) = \mathrm{Re}\,[F(i\omega)]$ and $\chi(\omega) = -\mathrm{Im}\,[F(i\omega)]$. The condition $F(\overline{i\omega}) = \overline{F(i\omega)}$ is equivalent to:

$$\varphi(-\omega) - i\chi(-\omega) = \varphi(\omega) + i\chi(\omega)$$

and hence $\varphi(\omega) = \varphi(-\omega)$ is an even function and $\chi(\omega) = -\chi(-\omega)$ is an odd function.

Using

$$\int_{-\infty}^{\infty} \varphi(\omega) \sin \omega t \, d\omega = 0$$

and

$$\int_{-\infty}^{\infty} \chi(\omega) \cos \omega t \, d\omega = 0,.$$

(3) reduces to

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) \cos \omega t \, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) \sin \omega t \, d\omega \tag{4}$$

for $t > 0$. Replacing t by $-t$,

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) \cos \omega t \, d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) \sin \omega t \, d\omega. \tag{5}$$

We therefore get the pair of formulas

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi(\omega) \sin \omega t \, d\omega \tag{6}$$

or

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\omega) \cos \omega t \, d\omega. \tag{7}$$

Since $\varphi(\omega)$ is even and $\chi(\omega)$ odd, these can be written

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \chi(\omega) \sin \omega t \, d\omega \tag{8}$$

or

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \varphi(\omega) \cos \omega t \, d\omega. \tag{9}$$

4. The Hurwitz and Zweifel Method for Trigonometric Integrals

The numerical evaluation of the integrals in (8) or (9) presents three difficulties. First the range is infinite. Second, for large t the integrands oscillate violently, and hence conventional methods of evaluation require an impractically small interval of integration. Third, there are strong cancellations from the positive and negative half-cycles of $\sin \omega t$ and $\cos \omega t$.

Hurwitz and Zweifel [2] have devised a procedure which largely circumvents these difficulties. They carry out the integration over successive half-cycles and then use a series-summing technique to reduce the number of half-cycles necessary. The integration over individual half-cycles is based on a Gaussian quadrature method. Details may be found in the article quoted.

The essential formulas are as follows, using the sine integral (8):

$$f(t) = \frac{2}{t} \sum_{n=0}^{\infty} I_n(t) = \frac{2}{t} \lim_{m\to\infty} S_m(t) \tag{10}$$

$$I_n(t) = (-1)^n \int_{-\frac{1}{2}}^{\frac{1}{2}} \chi\left[\frac{\pi}{t}(\omega + n + \tfrac{1}{2})\right] \cos \pi\omega \, d\omega \tag{11}$$

The Gaussian quadrature formula is:

$$I_n(t) = (-1)^n \sum_{j=i}^{N} \frac{W_j^N}{\cos \pi y_j^N} [\chi(\omega_{nj}^-) + \chi(\omega_{nj}^+)] \tag{12}$$

where

$$y_j^N = \frac{2j - 1}{2(2N + 1)}, \quad j = 1, 2, \cdots, N$$

$$\omega_{nj}^- = \frac{\pi}{t}(-y_j^N + n + \tfrac{1}{2})$$

$$\omega_{nj}^+ = \frac{\pi}{t}(y_j^N + n + \tfrac{1}{2})$$

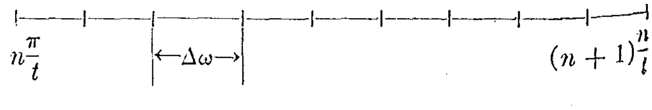
and the W_j^N are the solution of the system

$$2 \sum_{j=i}^{N} W_j^N \cos^{2\lambda-2}\left[\frac{(2j-1)\pi}{2(2N+1)}\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)},$$

$$\lambda = 1, \cdots, N.$$

The points ω_{nj} span a half-cycle with an ω increment of

$$\Delta\omega = \frac{\pi}{t(2N + 1)}.$$



The formulas for the cosine integral are very similar. Only the sine integral was programmed for the 650, but the program for the cosine integral would be almost the same. In general there is no reason for using $\chi(\omega)$ in preference to $\varphi($. In particular cases one or the other might be bette. ehaved, and hence one could achieve more accuracy with the cosine integral.

The convergence of the series (10) can be accelerated by applying an averaging process to the partial sums $S_m = \sum_{n=0}^{m} I_n$. We define a new sequence

$$S_m^1 = (S_m + S_{m+1})/2,$$

and in general $S_m^k = (S_m^{k-1} + S_{m+1}^{k-1})/2$. For relatively flat functions $\chi(\omega)$, the partial sums S_m oscillate about the limiting value, and hence the average can be expected to be more accurate than the individual terms. In the 650 program three averages were used. The computation is stopped if

$$\left| \frac{S_m^3 - S_{m+1}^3}{S_{m+1}^3} \right| < \epsilon,$$

where ϵ is an accuracy control constant which is fed into the program.

5. Modifications for Positive Poles and Small Negative Poles

Suppose that $F(s)$ has a pole at $s = a$ where $\mathrm{Re}(a) > 0$. Let $G(s) = F(s + a + \beta)$ where $\beta > 0$. Then $G(s)$ can be inverted by the above method; and

$$L^{-1}(G) = e^{-(a+\beta)t} L^{-1}(F)$$

or

$$L^{-1}(F) = e^{(a+\beta)t} L^{-1}(G).$$

If $F(s)$ has a pole at the origin it is of course easier to subtract off the singular part.

In one problem which was run using the 650 program, the function $\chi(\omega) = \mathrm{Im}\{F(i\omega)\}$ showed a sharp peak near $\omega = 0$. It was impossible to take a small enough interval of integration to adequately cover this peak. It was conjectured that it was due to a negative pole at $-a$, where $\chi(a) = \max$. The value of a was determined, and the function $2a\chi(a)\omega/(\omega^2 + a^2)$ was subtracted from $\chi(\omega)$. This corresponds to forming $G(s) = F(s) - 2a\chi(a)/(s + a)$, then inverting $G(s)$.

We have in this case

$$L^{-1}(F) = L^{-1}(G) + e^{at} 2a\chi(a).$$

6. A Program for Small t

The method described above will work only for t greater than some minimum value, which depends on the maximum number of points ($2N$) used per half-cycle. The $\Delta\omega$ associated with t and N is $\pi/t(2N + 1)$, so that if t is very small, N would have to be inordinately large. We decided to use an alternate integration technique for small t. A program using Simpson's rule has been written for t's up(he t such that the more efficient Gaussian integration can be used.

7. Sample Problems ($\epsilon = .001$)

(A) $$F(s) = \frac{1}{s^2 + s + 1}$$

$$f(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

Time	Analytical $f(t)$	Numerical $f(t)$
0.5	0.377	0.372
1.0	0.533	0.534
1.5	0.525	0.525
2.0	0.419	0.419
2.5	0.274	0.274
3.0	0.133	0.133
3.5	0.022	0.022
4.0	−0.0495	−0.0496
4.5	−0.0834	−0.0833
5.0	−0.0879	−0.0877
5.5	−0.0737	−0.0735
6.0	−0.0509	−0.0508
6.5	−0.0272	−0.0271
7.0	−0.0076	−0.0076
7.5	0.0057	0.0057
8.0	0.0127	0.0127
8.5	0.0145	0.0144
9.0	0.0128	0.0127
9.5	0.0093	0.0092
10.0	0.0054	0.0053

(B) $$F(s) = \frac{s + 1}{s^2 + s + 1}$$

$$f(t) = e^{-\frac{t}{2}} \left[\cos \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \right]$$

Time	Analytical $f(t)$	Numerical $f(t)$
0.5	0.896	0.888
1.0	0.660	0.665
1.5	0.389	0.388
2.0	0.151	0.151
2.5	−0.0233	−0.0233
3.0	−0.124	−0.124
3.5	−0.162	−0.162
4.0	−0.153	−0.153
4.5	−0.118	−0.118
5.0	−0.0746	−0.0743
5.5	−0.0336	−0.0336
6.0	−0.0023	−0.0023
6.5	0.0171	0.0171
7.0	0.0256	0.0256
7.5	0.0258	0.0257
8.0	0.0210	0.0209
8.5	0.0140	0.0139
9.0	0.0071	0.0071
9.5	0.0015	0.0015
10.0	−0.0022	0.0021

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tial fraction form) as

$$V_0(s) = \frac{K_0}{s} + \frac{K_{11}}{s+b-j\omega} + \frac{K_{12}}{s+b+j\omega} + \frac{K_{21}}{s+c-jf} + \frac{K_{22}}{s+c+jf}. \quad (16)$$

The time response of the system is then given by

$$v_0(t) = K_0 + 2\,|K_{11}|\,e^{-bt}\sin(\omega t + \phi_1) + 2\,|K_{21}|\,e^{-ct}\sin(ft + \phi_2), \qquad 0 \le t < \infty. \quad (17)$$

Now assume that the coefficients in (15) are arbitrarily assigned the following values: $a_0 = 16$, $a_1 = 32$, $b = 1.0$, $\omega = \sqrt{15}$, $c = 3$, and $f = \sqrt{27}$. The partial fraction expansion coefficients in (16) and (17) can be evaluated using these assumed values and both the single and multiple interval Laplace transform approximations can be made using (17).

The sample times for the values of m and n given in Table I were used in the Legendre–Gauss formula to approximate the transform of (17). Table II gives the actual value of the system transfer function (15) for each value of s along with the approximate values and the percent error for each of the a[illegible]ximate values.

Refer[illegible] to Table II, one sees that the multiple interval sampling results compare [illegible]ably with those obtained using the [illegible] interval technique. In general, the [illegible] of the approximate transform is [illegible]ly more accurate for fewer intervals, [illegible] the difference in accuracy is slight.

Conclusions

[illegible] principal advantage of the multiple [illegible] sampling technique is that more [illegible] of a function can be taken than [illegible] order of available roots and [illegible] would otherwise allow. To the best [illegible] authors' knowledge the highest order [illegible] roots and weights for the [illegible]-Gauss quadrature correspond to [illegible] formula [1]. Using the mul[illegible] sampling technique and the [illegible] weights for the 32nd-order for[illegible], the approximation could be [illegible] for 32, 64, 96, 128, etc. samples. [illegible] of these approximations could [illegible] compared for each value of s as a [illegible] whether the quadrature approxi[illegible] converging.

[illegible] of the multiple interval sampling [illegible] seems to be justified in many [illegible]. particularly where the [illegible] oximated is complex and [illegible] large sample size for satis[illegible] in the approximation of [illegible]. Finally, it should be noted that the technique is applicable to other Gauss quadrature approximation formulas; for example, see [4] and [5].

Wayne J. Felts
George E. Cook
Vanderbilt University
Nashville, Tenn.

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GAUSS QUAD

Numerical Inversion of the Laplace Transform

Abstract—An extension of Bellman's method for the numerical inversion of the Laplace transform is discussed. This extension is theoretically equivalent to the method of Lanczos. Tables of coefficients are given which facilitate the inversion of the Laplace transform with the aid of a desk computer.

Bellman, Kalaba, and Lockett [1] have outlined a method of numerical inversion of the Laplace transform. An extension of this method is presented in this correspondence. The method given here is based on Lagrange interpolation of the Laplace transform and is in this sense equivalent to the method of Lanczos [4], which can be considered as a Newton interpolation.

Bellman's method is as follows. Let $F(s)$ be a given Laplace transform and $f(t)$ the corresponding original function. Then

$$\int_0^\infty e^{-st} f(t)\, dt = F(s). \quad (1)$$

Substituting

$$e^{-t} = u \quad (2)$$

(1) takes the form

$$\int_0^1 u^{s-1} f(-\log u)\, du = F(s). \quad (3)$$

Applying the Gauss–Legendre quadrature formula, (3) yields

$$\sum_{i=1}^{N} w_i u_i^{s-1} f(-\log u_i) \approx F(s) \quad (4)$$

where u_i is the ith zero of the shifted Legendre polynomial P_N^* of degree N and w_i is the corresponding weight.

Letting s assume N different values, e.g., $s = 1, 2, \cdots, N$, a system of N linear equations is obtained with N unknowns $f(-\log u_i)$, $i = 1, 2, \cdots, N$. This system can be solved explicitly. The solution takes the form

$$f(t_i) \approx \sum_{k=1}^{N} a_{ik}^{(N)} F(k) \quad (5)$$

where

$$t_i = -\log u_i. \quad (6)$$

Equation (5) is the inversion formula given in [1]–[3]. In [1], [2], the coefficients $a_{ik}^{(N)}$ are tabulated for $N = 3(1)15$, however, with great roundoff errors.

Unfortunately, the inversion formula (5) gives only the values of $f(t)$ in a restricted number of nonequidistant points. To avoid this difficulty, several techniques are proposed in [1], [2], for instance, a change of t scale. The purpose of this correspondence is to present an extension of (5),

$$f(t) \approx \sum_{k=1}^{N} \varphi_k^{(N)}(e^{-t}) F(k) \quad (7)$$

where $\varphi_k^{(N)}(x)$ is a polynomial of degree $N - 1$.

Equation (5), if necessary after application of a change of t scale, gives the same result as (7). However, it is much easier to use (7) directly.

$F(s)$ can be approximated by an interpolating rational function

$$F(s) \approx \sum_{k=1}^{N} \frac{(-1)^{N-k}(k+N-1)!\prod\limits_{\substack{m=1\\m\neq k}}^{N}(s-m)}{((k-1)!)^2(N-k)!\prod\limits_{m=0}^{N-1}(s+m)} F(k). \quad (8)$$

Equation (8) is a generalized Lagrange interpolation in the points $s = 1, 2, \cdots, N$.

Inverting (8), the desired formula (7) is obtained, where

$$\varphi_k^{(N)}(e^{-t}) = \sum_{m=0}^{N-1} (-1)^{k+m+1} \frac{(N+k-1)!(N+m)!e^{-mt}}{((k-1)!)^2(N-k)!(m!)^2(N-1-m)!(k+m)} \quad (9)$$

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and, particularly,

$$\varphi_k^{(N)}(e^{-t})$$

$$= (-1)^{N+1} \frac{(2N-1)!}{((N-1)!)^2} P_{N-1}^*(e^{-t}). \tag{10}$$

The coefficients of $\varphi_k^{(N)}(x)$ are integers and are given in Table I, for $N = 3(1)7$. With the aid of this table and using (7), the inversion of the Laplace transform can be carried out very quickly. Tables II and III are interesting especially for calculations with a desk computer. Table II gives the values of $\varphi_k^{(6)}(e^{-t})$ for $k = 1(1)6$ and $t = 0.0(0.5)7.0$. Values of $\varphi_k^{(10)}(e^{-t})$ for $k = 1(1)10$ and $t = 0.0(0.5)10.0$ are listed in Table III. All the given figures are correct. Tables II and III are extensions of the tables given in [1], [2]. The advantage of the tables given here is that the t values are equidistant. This facilitates interpolation. Tables I–III were calculated using the IBM 1620 and IBM 360/40 of the Computing Centre of the University of Louvain.

Sometimes, particularly for digital computers, it is more convenient to apply a generalized Newton interpolation,

$$F(p) \approx \sum_{m=0}^{N-1} \frac{c_m}{p} \prod_{i=0}^{m} \frac{i-p}{i+p}. \tag{11}$$

Inversion of (11) gives

$$f(t) \approx \sum_{m=0}^{N-1} c_m P_m^*(e^{-t}) \tag{12}$$

where

$$c_m = (2m+1) \sum_{j=0}^{m} a_j^{(m)} F(j+1) \tag{13}$$

and $a_j^{(m)}$ is the coefficient of x^j in $P_m^*(x)$.

Equation (11) does not require the degree N to be chosen at the outset. Thus the truncation error can be estimated by adding one or more terms in (12). Moreover, (12) and (13) are more appropriate for programming on a digital computer. Equations (12) and (13) are equivalent to those given by Lanczos [4].

Conclusion

An extension of Bellman's method of numerical inversion of the Laplace transform is given. This generalization was inspired by the fact that, from the theoretical (but not from the numerical) point of view, Bellman's method is a special case of the method of Lanczos. However, Bellman's method and the extension of it presented here are more suitable for calculations with a desk computer, using the tables given here and in [1], [2].

R. Piessens
Dept. of Appl. Math.
University of Louvain
Heverlee, Belgium

TABLE I

Coefficients of $\varphi_k^{(N)}(x)$

N	k	1	x	x²	x³	x⁴	x⁵	x⁶
3	1	9	-36	30				
	2	-36	192	-180				
	3	30	-180	180				
4	1	16	-120	240	-140			
	2	-120	1200	-2700	1680			
	3	240	-2700	6480	-4200			
	4	-140	1680	-4200	2800			
5	1	25	-300	1050	-1400	630		
	2	-300	4800	-18900	26880	-12600		
	3	1050	-18900	79380	-117600	56700		
	4	-1400	26880	-117600	179200	-88200		
	5	630	-12600	56700	-88200	44100		
6	1	36	-630	3360	-7560	7560	-2772	
	2	-630	14700	-88200	211680	-220500	83160	
	3	3360	-88200	564480	-1411200	1512000	-582120	
	4	-7560	211680	-1411200	3628800	-3969000	1552320	
	5	7560	-220500	1512000	-3969000	4410000	-1746360	
	6	-2772	83160	-582120	1552320	-1746360	698544	
7	1	49	-1176	8820	-29400	48510	-38808	12012
	2	-1176	37632	-317520	1128960	-1940400	1596672	-504504
	3	8820	-317520	2857680	-10584000	18711000	-15717240	5045040
	4	-29400	1128960	-10584000	40320000	-72765000	62092800	-20180160
	5	48510	-1940400	18711000	-72765000	133402500	-115259760	37837800
	6	-38808	1596672	-15717240	62092800	-115259760	100590336	-33297264
	7	12012	-504504	5045040	-20180160	37837800	-33297264	11099088

TABLE II

Values of $\varphi_k^{(6)}(e^{-t})$ for $t = 0.0(0.5)7.0$

k	0.00	.50	1.00
1	-6.00000000	-1.30827892	2.36090779
2	210.00000000	55.93481856	-98.08826666
3	-1680.00000000	-513.07038085	818.53538201
4	5040.00000000	1650.11371119	-2240.43876680
5	-6300.00000000	-2071.26016793	2469.74350140
6	2772.00000000	882.37761424	-954.00546228
k	**1.50**	**2.00**	**2.50**
1	-4.06524843	-4.04979937	3.07752813
2	109.77595262	198.49752376	89.73867789
3	-467.37987000	-1254.98392699	-790.49788528
4	745.40459774	2974.66310326	2139.78690301
5	-488.34752235	-3026.24577537	-2353.45258685
6	103.73350290	1114.27435660	913.77731780
k	**3.00**	**3.50**	**4.00**
1	12.07537841	19.83761304	25.54269674
2	-91.96200963	-260.87880139	-389.07194785
3	202.94266033	1173.71313114	1925.42083508
4	-95.17046307	-2358.00279830	-4134.49775528
5	-133.42703551	2174.57351039	4004.72742020
6	106.41657502	-750.29955139	-1434.80827837
k	**4.50**	**5.00**	**5.50**
1	29.40573934	31.90534008	33.48093780
2	-477.29563513	-534.89215284	-571.38316284
3	2447.93710332	2790.91184606	3008.87863338
4	-5377.68877713	-6196.67786093	-6718.23507994
5	5291.68741923	6141.72231963	6683.85018938
6	-1917.91347467	-2237.62926619	-2441.76101238
k	**6.00**	**6.50**	**7.00**
1	34.45891583	35.06040236	35.42830257
2	-594.10104697	-608.09808670	-616.66851598
3	3144.82090749	3228.66778601	3280.04032275
4	-7043.91333486	-7244.92945712	-7368.14353004
5	7022.66471265	7231.89581080	7360.18429155
6	-2569.42206194	-2648.29451338	-2696.65076939

TABLE III

VALUES OF $\varphi_k^{(10)}(e^{-t})$ FOR $t = 0.0(0.5)10.0$

k	0.00	.50	1.00
1	-10.00000000	-3.22210553	-1.20724842
2	990.00000000	323.11149776	65.08745345
3	-23760.00000000	-7749.54867095	-219.40431994
4	240240.00000000	77207.67278480	-10190.79610533
5	-1261260.00000000	-393822.28456122	105478.81020693
6	3783780.00000000	1132993.65156535	-424901.39844463
7	-6726720.00000000	-1910432.53598472	877830.26850133
8	7001280.00000000	1870608.85838105	-987677.16126394
9	-3938220.00000000	-984472.77333481	576511.26149675
10	923780.00000000	215348.02820457	-136898.50725871

k	1.50	2.00	2.50
1	1.91442141	4.10058030	-11.10826641
2	-81.94729694	-569.01678767	852.93270486
3	-624.36193225	15312.71897637	-11408.19329376
4	26585.76054532	-145001.25562383	65124.48766667
5	-199496.01639921	680964.32401105	-198446.80900003
6	685688.73918643	-1811150.91033552	347048.88539554
7	-1278806.43431308	2862407.59054988	-348575.21202372
8	1338197.52155336	-2665740.04290188	185621.66866344
9	-740010.82551662	1351645.62011152	-39091.69599096
10	168548.54820115	-287876.09090456	-1125.96449644

k	3.00	3.50	4.00
1	-10.53997611	8.15189809	32.48724445
2	1482.11478782	702.68838242	-669.08472279
3	-27989.70574588	-18326.46675031	3983.27629269
4	222836.35758119	166256.78749296	-1913.88661121
5	-949834.76526399	-765357.80071197	-66747.94481010
6	2380422.38225424	2021307.94896414	294635.64905378
7	-3618428.23747045	-3193277.79437983	-589957.16842634
8	3280855.72644559	2992780.24072674	634349.31058481
9	-1632101.80877851	-1519467.28015852	-355098.54361429
10	342771.82802222	325377.41028076	81387.40859872

k	4.50	5.00	5.50
1	53.99849444	70.06422447	81.05295801
2	-1984.28188949	-3001.56211546	-3709.67846732
3	26565.39982483	44420.47159784	56989.32555406
4	-178835.00065276	-320838.31664003	-421539.26497989
5	629663.59436219	1303432.09920694	1740995.36533852
6	-1616932.34637402	-3140587.05351754	-4330255.69161742
7	2343541.83957117	4759082.46151701	6493422.53263611
8	-2050273.91287674	-4272375.66729870	-5871810.12750531
9	991926.36029443	2111635.37646371	2919231.99809773
10	-203727.27428651	-441572.13063050	-613411.64699217

	6.00	6.50	7.00
1	88.20774586	92.73496592	95.55158768
2	-4175.60667315	-4471.95792073	-4656.94212484
3	65302.84925067	70610.87251234	73930.75434469
4	-488413.05384027	-531207.66529755	-558008.88582154
5	2032405.71724061	2219191.25838511	2336281.39989085
6	-5047361.17948981	-5526818.05619971	-5827588.72868261
7	7652888.31856316	6397684.61407043	8865165.36880940
8	-6942519.84688508	-7630824.93198598	-8063039.57225576
9	3460454.35849763	3808595.07508060	4027285.93751891
10	-728677.45341986	-802860.50810107	-849474.00007946

	7.50	8.00	8.50
1	97.28635845	98.34835014	98.99610535
2	-4771.09819697	-4841.06434071	-4883.76993669
3	75981.91016653	77239.94867725	78008.15020387
4	-574580.77701925	-554749.62433948	-590960.83021483
5	2408721.97287699	2453187.92108452	2480353.54018069
6	-6013746.05018261	-6128043.18467047	-6197881.42479234
7	9154600.20219578	9332342.67744512	9440960.33263047
8	-8330710.05672540	-8495112.80748345	-8595588.10110982
9	4162750.24632353	4245962.87765907	4296822.48578894
10	-878353.09547124	-896094.76293878	-906939.17840354

	9.00	9.50	10.00
1	99.39032656	99.62992693	99.77543353
2	-4909.77148907	-4923.57883504	-4935.17995932
3	78475.99409823	79760.45852817	72933.25380378
4	-594744.17523523	-597044.81085080	-598442.39804097
5	2496902.57439769	2506966.71299404	2513080.73637043
6	-6240430.10600054	-6266307.11686555	-6282028.08018420
7	9507139.93798699	9547390.40122411	9571844.25178973
8	-8656810.13887031	-8694046.69049547	-8716669.93484295
9	4327813.92565762	4346664.11306742	4358116.82971331
.0	-913547.50905764	-917567.04472352	-920009.20918637

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Procedures to Check the Adjoint Equations When Using the Method of Steepest Ascent

Abstract—The method of steepest ascent is well documented in the literature [1]. However, its application to problems of high order (over 20) is not straightforward [2]. One problem that arises in the application of the method of steepest ascent, particularly to problems of high order and generally to problems of any order, is the pinpointing of errors in programming or in deriving the adjoint equations. This correspondence presents a systematic method for pinpointing such errors. First, a derivation of equations pertinent to the method of steepest ascent as developed by Bryson *et al.* [1] is presented. Then checks on the adjoints are followed by an illustrative example of the use of these checks.

I. Total Variation of the Cost Functional

An expression is derived for the total variation of the payoff function which is fundamental to the method of steepest ascent. For a complete mathematical description of steepest ascent, see Bryson *et al.* [1].

Let the system be represented by the set of first-order differential equations with x a state vector and u the control vector,

$$\dot{x} = f(x, u, t). \tag{1}$$

Define a payoff functional in terms of the final state $x(T)$ and final time T as

$$\Phi = \Phi(x(T), T). \tag{2}$$

Such a formulation causes no loss in generality.

One now forms the incremental system equations

$$\delta\dot{x} = \left(\frac{\partial f}{\partial x}\right)_0 \delta x + \left(\frac{\partial f}{\partial u}\right)_0 \delta u \tag{3}$$

where the subscript zero indicates evaluation along a trajectory about which the perturbations are taken.

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[GAUSSIAN QUADRATURE]

On the Condition of a Matrix Arising in the Numerical Inversion of the Laplace Transform

By Walter Gautschi

Abstract. Bellman, Kalaba, and Lockett recently proposed a numerical method for inverting the Laplace transform. The method consists in first reducing the infinite interval of integration to a finite one by a preliminary substitution of variables, and then employing an n-point Gauss-Legendre quadrature formula to reduce the inversion problem (approximately) to that of solving a system of n linear algebraic equations. Luke suggests the possibility of using Gauss-Jacobi quadrature (with parameters α and β) in place of Gauss-Legendre quadrature, and in particular raises the question whether a judicious choice of the parameters α, β may have a beneficial influence on the condition of the linear system of equations. The object of this note is to investigate the condition number cond (n, α, β) of this system as a function of n, α, and β. It is found that cond (n, α, β) is usually larger than cond (n, β, α) if $\beta > \alpha$, at least asymptotically as $n \to \infty$. Lower bounds for cond (n, α, β) are obtained together with their asymptotic behavior as $n \to \infty$. Sharper bounds are derived in the special cases $\alpha = \beta$, n odd, and $\alpha = \beta = \pm\frac{1}{2}$, n arbitrary. There is also a short table of cond (n, α, β) for $\alpha, \beta = -.8(.2)0, .5, 1, 2, 4, 8, 16, \beta \leqq \alpha$, and $n = 5, 10, 20, 40$. The general conclusion is that cond (n, α, β) grows at a rate which is something like a constant times $(3 + \sqrt{8})^n$, where the constant depends on α and β, varies relatively slowly as a function of α, β, and appears to be smallest near $\alpha = \beta = -1$. For quadrature rules with equidistant points the condition grows like $(2\sqrt{2}/3\pi)8^n$.

1. In [4], Bellman, Kalaba, and Lockett propose a numerical procedure to invert the Laplace transform

$$\int_0^\infty e^{-st}u(t)dt = F(s)\,. \tag{1.1}$$

Briefly, the procedure consists of first substituting $x = e^{-t}$, to bring (1.1) into the form

$$\int_0^1 x^{s-1}g(x)dx = F(s)\,, \qquad g(x) = u(-\ln x)\,, \tag{1.2}$$

and then employing Gaussian quadrature to approximate (1.2) by

$$\sum_{i=1}^n w_i x_i^k g(x_i) = F(k+1)\,, \qquad (k = 0, 1, 2, \cdots, n-1)\,, \tag{1.3}$$

where x_i are the zeros of the shifted Legendre polynomical $p_n(x) = P_n(2x - 1)$ and w_i the associated weight factors. Letting $y_i = w_i g(x_i)$, the method thus boils down to solving the system of linear algebraic equations

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$$\sum_{i=1}^{n} x_i^k y_i = F(k+1)\,, \qquad (k = 0, 1, 2, \cdots, n-1)\,. \tag{1.4}$$

In reviewing the work of Bellman et al., Y. L. Luke [8] generalizes their approach by employing the substitution $x = e^{-vt}$ $(v > 0)$ in (1.1), and by using Jacobi polynomials in place of Legendre polynomials. This again leads to a system of equations (1.4) where now x_i are the zeros of the shifted Jacobi polynomial $p_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$, and $F(k+1)$ on the right must be replaced by $F((k+1)v)$.

The system (1.4) can be solved analytically in a number of ways, the coefficient matrix being a Vandermonde matrix. However, as noted in [4], the ill-conditioned character of the system may well require high-precision calculations, especially if n is fairly large. Luke [8] raises the question of whether or not "the detrimental effects of ill-conditioning can be removed or mitigated by the use of other choices of α and β" (other than $\alpha = \beta = 0$). The purpose of this note is to give a detailed answer to this question.

We first obtain a closed expression for the condition number of the coefficient matrix in (1.4). In Section 3 we compare the condition number for $p_n^{(\alpha,\beta)}$ with that for $p_n^{(\beta,\alpha)}$ and find that the former is usually larger than the latter if $\beta > \alpha$, at least asymptotically as $n \to \infty$. Section 4 contains a short table of the condition number for $p_n^{(\alpha,\beta)}$, where $\alpha, \beta = -.8(.2)0, .5, 1, 2, 4, 8, 16, \beta \leqq \alpha$, and $n = 5, 10, 20, 40$. Section 5 exhibits lower bounds for the condition number, together with their asymptotic behavior. Sharper results are obtained in Section 6 in the case $\alpha = \beta$, n odd, and in Section 7 for general n, and $\alpha = \beta = \pm\frac{1}{2}$. For comparison we consider in Section 8 the case of equidistant abscissas x_i.

The general conclusion is that the condition number grows at a rate which is something like a constant times $(3 + \sqrt{8})^n$ [$(2\sqrt{2/3\pi})8^n$ for equidistant abscissas], where the constant depends on α and β and varies relatively slowly as a function of α and β. As expected, there is no escape from ill-conditioning, which, after all, only reflects the fact that the original inversion problem (1.1) is not well posed (cf., in this connection, [1], [2], [3], [9], [11], [13], [14]).

2. Let $p_n(x)$ be an arbitrary polynomial of degree n whose zeros x_i are distinct and located in the interval [0, 1]. Let

$$V(p_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} \tag{2.1}$$

denote the Vandermonde matrix of the zeros x_i. We shall consider the condition number

$$\operatorname{cond}_\infty [V(p_n)] = \|V(p_n)\|_\infty \|[V(p_n)]^{-1}\|_\infty\,, \tag{2.2}$$

where $\|\cdot\|_\infty$ denotes the ∞-matrix norm ("maximum row sum"). Clearly,

$$\|V(p_n)\|_\infty = n\,. \tag{2.3}$$

In [5] we have shown that under the assumptions made,

$$\|[V(p_n)]^{-1}\|_\infty = \max_i \prod_{j \neq i} \left(\frac{1 + x_j}{|x_i - x_j|}\right). \tag{2.4}$$

Combining (2.3) and (2.4), and rewriting (2.4) in terms of p_n and its derivative, we obtain

$$\text{cond}_\infty\,[V(p_n)] = \frac{n|p_n(-1)|}{\min_i\,\{(1+x_i)|p_n'(x_i)|\}}\,. \tag{2.5}$$

3. We now let p_n be the shifted Jacobi polynomial $p_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$, $\alpha > -1$, $\beta > -1$. We first show that

$$\text{cond}_\infty\,[V(p_n^{(\alpha,\beta)})] = \gamma_n \frac{P_n^{(\beta,\alpha)}(3)}{P_n^{(\alpha,\beta)}(3)}\,\text{cond}_\infty\,[V(p_n^{(\beta,\alpha)})]\,, \qquad \tfrac{1}{2} < \gamma_n < 2\,, \tag{3.1}$$

where the constant γ_n depends on α and β. Indeed, it is well known that

$$p_n^{(\alpha,\beta)}(x) = (-1)^n p_n^{(\beta,\alpha)}(1-x)\,,$$
$$p_n^{(\alpha,\beta)\prime}(x) = (-1)^{n+1} p_n^{(\beta,\alpha)\prime}(1-x)\,.$$

In particular, if x_i is a zero of $p_n^{(\alpha,\beta)}$ then $\xi_i = 1 - x_i$ is a zero of $p_n^{(\beta,\alpha)}$. Therefore,

$$\begin{aligned}(1+x_i)|p_n^{(\alpha,\beta)\prime}(x_i)| &= (1+x_i)|p_n^{(\beta,\alpha)\prime}(\xi_i)| \\ &= \frac{1+x_i}{2-x_i}(1+\xi_i)|p_n^{(\beta,\alpha)\prime}(\xi_i)|\,,\end{aligned} \tag{3.2}$$

and since $\frac{1}{2} < (1+x)/(2-x) < 2$ for $0 < x < 1$, it follows that

$$\min_i\,\{(1+x_i)|p_n^{(\alpha,\beta)\prime}(x_i)|\} = \frac{1}{\gamma_n}\min_i\,\{(1+\xi_i)|p_n^{(\beta,\alpha)\prime}(\xi_i)|\}\,,\ \tfrac{1}{2} < \gamma_n < 2\,.$$

Consequently, by (2.5),

$$\text{cond}_\infty\,[V(p_n^{(\alpha,\beta)})] = \gamma_n|p_n^{(\alpha,\beta)}(-1)/p_n^{(\beta,\alpha)}(-1)|\,\text{cond}_\infty\,[V(p_n^{(\beta,\alpha)})]\,,$$

which is equivalent to (3.1).

Noting that [10, p. 194]

$$P_n^{(\beta,\alpha)}(3) \sim \frac{n^{-1/2}}{\pi^{1/2}2^{(2\alpha+5)/4}}(3+\sqrt{8})^{n+(\alpha+\beta+1)/2} \quad (n\to\infty)\,, \tag{3.3}$$

we obtain from (3.1),

$$\text{cond}_\infty\,[V(p_n^{(\alpha,\beta)})] \sim \gamma_n\cdot 2^{(\beta-\alpha)/2}\,\text{cond}_\infty\,[V(p_n^{(\beta,\alpha)})]\,, \qquad (n\to\infty)\,. \tag{3.4}$$

Our computations (cf. Section 4) have revealed that in most cases the minimum in (2.5) is assumed for x_i near $\frac{1}{2}$ (though not necessarily closest to $\frac{1}{2}$), so that in these cases $\gamma_n \approx 1$. Taking this into account it appears from (3.4) that for n sufficiently large the condition number for $p_n^{(\alpha,\beta)}$ is greater than that for $p_n^{(\beta,\alpha)}$ if $\beta > \alpha$. As was observed by computation this remains generally true for smaller values of n as well (typically for those of Table 1), although in a few instances in the region $-1 < \alpha, \beta < 0$, $\beta > \alpha$, it was found that $\text{cond}_\infty\,[V(p_n^{(\alpha,\beta)})]$ is slightly less than $\text{cond}_\infty\,[V(p_n^{(\beta,\alpha)})]$.

4. In order to compute the condition number in (2.5) for $p_n(x) = p_n^{(\alpha,\beta)}(x)$, we make use of the fact that these polynomials satisfy the orthogonality relation

$$\int_0^1 p_n(x)p_m(x)(1-x)^\alpha x^\beta dx = h_n\delta_{n,m}\,, \tag{4.1}$$

where $h_n = \Gamma(n+\alpha+1)\Gamma(n+\beta+1)/((2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1))$ and $\delta_{n,m}$ is the Kronecker delta. With $p_r^*(x) = h_r^{-1/2}\,p_r(x)$ denoting the normalized polynomials, we may compute $p_n^*(x)$ from the recurrence relation

$$\begin{aligned} p_{r+1}^*(x) &= ((x-a_r)p_r^*(x) - b_r p_{r-1}^*(x))/b_{r+1}\,, \qquad (r = 0, 1, 2, \cdots, n-1)\,, \\ p_{-1}^*(x) &= 0\,, \qquad p_0^*(x) = \left\{\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}^{1/2}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2}\left(1 - \frac{\alpha-\beta}{\alpha+\beta+2}\right), \\ a_r &= \frac{1}{2}\left\{1 - \frac{\alpha^2-\beta^2}{(2r+\alpha+\beta)(2r+\alpha+\beta+2)}\right\}, \qquad (r \geqq 1)\,, \\ b_1 &= \frac{1}{\alpha+\beta+2}\left\{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+3}\right\}^{1/2}, \\ b_r &= \frac{1}{2r+\alpha+\beta}\left\{\frac{r(r+\alpha)(r+\beta)(r+\alpha+\beta)}{(2r+\alpha+\beta-1)(2r+\alpha+\beta+1)}\right\}^{1/2}, \qquad (r \geqq 2)\,. \end{aligned} \tag{4.3}$$

The zeros of $p_n^*(x)$ may now be computed from (4.2) by a combination of Newton's method and successive deflation as described in [6, p. 261]. Hence the condition number of $V(p_n)$ can be computed directly from (2.5) for any value of α and β. Selected results* are shown in Table 1. (The numbers in parentheses denote the powers of 10 by which the preceding numbers are to be multiplied.) For reasons indicated at the end of Section 3 we restrict our tabulation to the region $\beta \leqq \alpha$.

The results in Table 1 indicate that $\mathrm{cond}_\infty\,[V(p_n^{(\alpha,\beta)})]$ for fixed α is an increasing function of β, if $-1 < \alpha \leqq 0$, and changes from a decreasing to an increasing function as β varies from -1 to α, if $\alpha > 0$. There is thus a "valley" of low condition number extending approximately (and more or less independently of n) along the line $\beta = -1 + 2\alpha/7$, as was determined by additional calculations. Along this valley, as well as along the diagonal $\alpha = \beta$, and near the line $\beta = -1$, the condition number increases with α and thus appears to be smallest near $\alpha = \beta = -1$.

5. A lower bound for the condition number in (2.5) may be obtained as follows. Let

$$\max_{0 \leqq x \leqq 1} |p_n(x)| = \mu_n\,. \tag{5.1}$$

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* In the range $-1 < \alpha \leqq 3$, $-1 < \beta \leqq 3$, $\beta \leqq \alpha$, and for $n = 5$ and $n = 8$, the zeros of $p_n^{(\alpha,\beta)}(x)$ as computed were checked against those tabulated in [7]. Disagreement never exceeded one unit of the last (eighth) significant digit. For $n = 40$, successive deflation was used only for the first 20 zeros. The remaining zeros were obtained from the original polynomial by Newton's method and a simple search procedure.

TABLE 1.

Selected values of $\mathrm{cond}_\infty\,[V(p_n^{(\alpha,\beta)})]$

α	β	n = 5	10	20	40	α	β	n = 5	10	20	40
−.8	−.8	9.82(2)	6.66(6)	2.99(14)	6.12(29)	2.0	.5	2.39(3)	2.05(7)	1.04(15)	2.24(30)
−.6	−.8	1.05(3)	7.15(6)	3.23(14)	6.60(29)		1.0	2.71(3)	2.32(7)	1.20(15)	2.62(30)
	−.6	1.08(3)	7.48(6)	3.39(14)	6.94(29)		2.0	3.54(3)	3.20(7)	1.72(15)	3.89(30)
−.4	−.8	1.13(3)	7.68(6)	3.49(14)	7.13(29)	4.0	−.8	4.20(3)	3.85(7)	2.11(15)	4.82(30)
	−.6	1.15(3)	8.00(6)	3.63(14)	7.44(29)		−.6	3.98(3)	3.64(7)	1.99(15)	4.57(30)
	−.4	1.19(3)	8.39(6)	3.84(14)	7.90(29)		−.4	3.83(3)	3.47(7)	1.93(15)	4.42(30)
−.2	−.8	1.21(3)	8.24(6)	3.76(14)	7.73(29)		−.2	3.73(3)	3.38(7)	1.90(15)	4.34(30)
	−.6	1.23(3)	8.57(6)	3.88(14)	7.99(29)		0	3.66(3)	3.40(7)	1.88(15)	4.33(30)
	−.4	1.25(3)	8.94(6)	4.10(14)	8.43(29)		.5	3.60(3)	3.52(7)	1.95(15)	4.48(30)
	−.2	1.34(3)	9.53(6)	4.39(14)	9.03(29)		1.0	3.64(3)	3.70(7)	2.10(15)	4.88(30)
0	−.8	1.30(3)	8.85(6)	4.07(14)	8.38(29)		2.0	4.32(3)	4.47(7)	2.65(15)	6.36(30)
	−.6	1.31(3)	9.17(6)	4.18(14)	8.61(29)		4.0	6.44(3)	7.67(7)	5.20(15)	1.36(31)
	−.4	1.33(3)	9.54(6)	4.38(14)	9.02(29)	8.0	−.8	1.03(4)	1.37(8)	1.01(16)	2.79(31)
	−.2	1.38(3)	9.98(6)	4.63(14)	9.58(29)		−.6	9.52(3)	1.26(8)	9.20(15)	2.54(31)
	0	1.50(3)	1.08(7)	5.00(14)	1.03(30)		−.4	8.91(3)	1.18(8)	8.58(15)	2.36(31)
.5	−.8	1.53(3)	1.06(7)	4.99(14)	1.03(30)		−.2	8.44(3)	1.10(8)	8.03(15)	2.22(31)
	−.6	1.53(3)	1.09(7)	5.05(14)	1.05(30)		0	8.07(3)	1.04(8)	7.71(15)	2.13(31)
	−.4	1.54(3)	1.12(7)	5.19(14)	1.08(30)		.5	7.45(3)	9.81(7)	7.23(15)	2.00(31)
	−.2	1.56(3)	1.16(7)	5.44(14)	1.13(30)		1.0	7.12(3)	9.75(7)	7.11(15)	1.97(31)
	0	1.62(3)	1.21(7)	5.74(14)	1.20(30)		2.0	6.95(3)	1.01(8)	7.54(15)	2.15(31)
	.5	1.93(3)	1.45(7)	6.87(14)	1.44(30)		4.0	8.68(3)	1.31(8)	1.09(16)	3.34(31)
1.0	−.8	1.80(3)	1.29(7)	6.13(14)	1.28(30)		8.0	1.48(4)	2.99(8)	3.43(16)	1.35(32)
	−.6	1.77(3)	1.28(7)	6.09(14)	1.28(30)	16.0	−.8	3.96(4)	1.10(9)	1.66(17)	7.77(32)
	−.4	1.77(3)	1.32(7)	6.23(14)	1.30(30)		−.6	3.54(4)	9.83(8)	1.44(17)	6.74(32)
	−.2	1.79(3)	1.36(7)	6.40(14)	1.35(30)		−.4	3.22(4)	8.87(8)	1.26(17)	5.93(32)
	0	1.81(3)	1.41(7)	6.73(14)	1.41(30)		−.2	2.96(4)	8.07(8)	1.15(17)	5.35(32)
	.5	2.05(3)	1.61(7)	7.81(14)	1.65(30)		0	2.76(4)	7.40(8)	1.06(17)	4.86(32)
	1.0	2.41(3)	1.92(7)	9.38(14)	2.02(30)		.5	2.38(4)	6.15(8)	8.69(16)	4.05(32)
2.0	−.8	2.44(3)	1.89(7)	9.24(14)	1.99(30)		1.0	2.14(4)	5.44(8)	7.78(16)	3.56(32)
	−.6	2.37(3)	1.81(7)	9.08(14)	1.94(30)		2.0	1.86(4)	4.89(8)	6.67(16)	3.11(32)
	−.4	2.33(3)	1.82(7)	9.00(14)	1.94(30)		4.0	1.75(4)	4.63(8)	6.56(16)	3.18(32)
	−.2	2.31(3)	1.85(7)	9.16(14)	1.96(30)		8.0	2.20(4)	6.48(8)	1.10(17)	6.28(32)
	0	2.32(3)	1.90(7)	9.37(14)	2.01(30)		16.0	4.20(4)	1.98(9)	6.38(17)	6.73(33)

Since $|\mu_n^{-1} p_n(x)| \leq 1$ on $[0, 1]$ it follows from a theorem of Markov (see, e.g., [12, p. 36]) that $|\mu_n^{-1} p_n'(x)| \leq 2n^2$ on $[0, 1]$, and so

$$(5.2) \qquad (1 + x_i)|p_n'(x_i)| \leqq 4n^2\mu_n\,, \qquad (i = 1, 2, \cdots, n)\,.$$

Consequently, by (2.5),

$$(5.3) \qquad \operatorname{cond}_\infty [V(p_n)] \geqq \kappa_n\,, \qquad \kappa_n = |p_n(-1)|/4n\mu_n\,.$$

If $p_n(x) = p_n^{(\alpha,\beta)}(x)$, we may take advantage of known asymptotic results for Jacobi polynomials to obtain an asymptotic expression for κ_n in (5.3). As $n \to \infty$, we have [10, p. 380]

$$(5.4) \qquad \begin{aligned} \mu_n &\sim n^q/\Gamma(q+1) \quad \text{if} \quad q \geqq -\tfrac{1}{2} \\ &\sim \pi^{-1/2}|\alpha + \tfrac{1}{2}|^{-\alpha/2-1/4}|\beta + \tfrac{1}{2}|^{-\beta/2-1/4}|\alpha + \beta + 1|^{(\alpha+\beta+1)/2}n^{-1/2} \\ &\qquad\qquad \text{if} \quad -1 < q < -\tfrac{1}{2}\,, \end{aligned}$$

where $q = \max(\alpha, \beta)$. Combining (5.4) with (3.3) we obtain from (5.3)

$$(5.5) \qquad \kappa_n \sim \frac{\Gamma(q+1)}{\sqrt{\pi}\,2^{(2\alpha+13)/4}}\, n^{-(q+3/2)}(3 + \sqrt{8})^{n+(\alpha+\beta+1)/2}\,, \qquad (q \geqq -\tfrac{1}{2}, n \to \infty)\,,$$

and

$$(5.6) \qquad \begin{aligned} \kappa_n \sim \frac{|\alpha + \frac{1}{2}|^{\alpha/2+1/4}|\beta + \frac{1}{2}|^{\beta/2+1/4}}{2^{(2\alpha+13)/4}|\alpha + \beta + 1|^{(\alpha+\beta+1)/2}}\, n^{-1}(3 + \sqrt{8})^{n+(\alpha+\beta+1)/2}\,, \\ (-1 < q < -\tfrac{1}{2}, n \to \infty)\,. \end{aligned}$$

The powers of n appearing in (5.5), (5.6) are due to the crudeness of the inequality (5.2) and do not reflect the true asymptotic behavior of $\operatorname{cond}_\infty [V(p_n^{(\alpha,\beta)})]$. In fact, if x_i is restricted to a closed interval in the interior of [0, 1] (e.g., i such that x_i is the smallest zero of $p_n^{(\alpha,\beta)}$ larger than or equal to $\frac{1}{2}$), then it is known [10, p. 237] that

$$(5.7) \qquad |p_n^{(\alpha,\beta)\prime}(x_i)| \mathrel{\dot{\sim}} n^{1/2}\,, \qquad (n \to \infty)\,,$$

the symbol $\dot{\sim}$ meaning that the ratio of the left-hand and right-hand expression remains between certain positive bounds depending only on α and β. It thus follows from (2.5) and (3.3) that

$$(5.8) \qquad \operatorname{cond}_\infty [V(p_n^{(\alpha,\beta)})] \geqq \kappa_n'\,, \qquad \kappa_n' \mathrel{\dot{\sim}} (3 + \sqrt{8})^n\,, \qquad (n \to \infty)\,.$$

If, as all numerical evidence indicates, the points at which the minimum in (2.5) is assumed remain in a closed interval inside the open interval (0, 1) as $n \to \infty$, then inequality in (5.8) may be replaced by equality.

6. Considerably sharper bounds can be had if $\alpha = \beta$. We thus consider

$$(6.1) \qquad \begin{aligned} p_n(x) = C_n^{(\sigma)}(2x - 1) = \frac{\Gamma(\sigma + \frac{1}{2})\Gamma(n + 2\sigma)}{\Gamma(2\sigma)\Gamma(n + \sigma + \frac{1}{2})} P_n^{(\sigma-1/2,\sigma-1/2)}(2x - 1)\,, \\ \sigma > -\tfrac{1}{2}\,, \end{aligned}$$

and for convenience we assume that n is odd. Then, by symmetry, $x_i = \frac{1}{2}$ for some $i = i_0$, so that for this zero,

$$p_n'(x_{i_0}) = p_n'(\tfrac{1}{2}) = 2C_n^{(\sigma)\prime}(0) = 2(n + 2\sigma - 1)C_{n-1}^{(\sigma)}(0)\,.$$

Since

$$C_{n-1}^{(\sigma)}(0) = (-1)^{(n-1)/2} \frac{\Gamma((n+2\sigma-1)/2)}{\Gamma(\sigma)\Gamma((n+1)/2)}, \qquad (\sigma \neq 0)$$
$$= 2(-1)^{(n-1)/2}/(n-1), \qquad (\sigma = 0),$$

we obtain

$$(1 + x_{i_0})|p_n'(x_{i_0})| = 6 \frac{\Gamma((n+2\sigma+1)/2)}{|\Gamma(\sigma)|\Gamma((n+1)/2)}, \qquad (\sigma \neq 0)$$
$$= 6, \qquad (\sigma = 0).$$

Hence, from (2.5),

$$\text{(6.2)} \qquad \operatorname{cond}_\infty [V(p_n)] \geqq \kappa_n, \qquad \kappa_n = \frac{n}{6} |C_n^{(\sigma)}(3)| \frac{|\Gamma(\sigma)|\Gamma((n+1)/2)}{\Gamma((n+2\sigma+1)/2)}, \qquad (\sigma \neq 0)$$
$$= \frac{n}{6} C_n^{(0)}(3), \qquad (\sigma = 0).$$

From the known asymptotic behavior of $P_n^{(\sigma-1/2,\sigma-1/2)}(x)$ as $n \to \infty$ [10, p. 194] and from Stirling's formula we find

$$\text{(6.3)} \qquad C_n^{(\sigma)}(3) \sim \frac{\Gamma(\sigma+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(2\sigma)2^{(\sigma+2)/2}} n^{\sigma-1}(3+\sqrt{8})^{n+\sigma}, \qquad (\sigma \neq 0, n \to \infty).$$

Furthermore,

$$C_n^{(0)}(3) = \frac{2}{n} T_n(3) \sim \frac{1}{n}(3+\sqrt{8})^n, \qquad (n \to \infty),$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind. Substituting in (6.2), and using Stirling's formula and the duplication formula for the gamma function, we obtain

$$\text{(6.4)} \qquad \kappa_n \sim \frac{1}{6 \cdot 8^{\sigma/2}}(3+\sqrt{8})^{n+\sigma}, \qquad (\sigma > -\tfrac{1}{2}, n \overset{\text{(odd)}}{\to} \infty),$$

a result which obviously improves upon (5.5), (5.6) and is more precise than (5.8).

The case $p_n(x) = P_n(2x-1)$ originally considered in [4] corresponds to $\sigma = \frac{1}{2}$, in which case (6.4) gives

$$\text{(6.5)} \qquad \kappa_n \sim \frac{1}{6 \cdot 8^{\frac{1}{4}}}(3+\sqrt{8})^{n+1/2}, \qquad (\sigma = \tfrac{1}{2}, n \overset{\text{(odd)}}{\to} \infty).$$

The corresponding analysis for even n appears to be rather more difficult, for general $\sigma > -\frac{1}{2}$, and we shall not pursue this any further. If $\sigma = 0$, or $\sigma = 1$, then (6.4) remains valid for general n, as will be seen in the next section.

7. The cases $\alpha = \beta = \pm\frac{1}{2}$ merit special attention since the Jacobi polynomials then reduce to Chebyshev polynomials (of the first and second kind), the zeros and weight factors of which are known explicitly.

We begin with $\alpha = \beta = -\frac{1}{2}$, or, equivalently $p_n(x) = T_n(2x-1)$. We have

$$|p_n(-1)| = T_n(3) = \tfrac{1}{2}[(3+\sqrt{8})^n + (3-\sqrt{8})^n],$$

so that

$$|p_n(-1)| > \tfrac{1}{2}(3+\sqrt{8})^n . \tag{7.1}$$

Since the zeros x_i of $p_n(x)$ satisfy

$$2x_i - 1 = \cos\theta_i, \quad \theta_i = \frac{2i-1}{2n}\pi , \qquad (i = 1, 2, \cdots, n) ,$$

and $T_n{}'(\cos\theta) = n(\sin n\theta)/\sin\theta$, we get

$$p_n{}'(x_i) = 2T_n{}'(\cos\theta_i) = (-1)^{i-1}\, 2n/\sin\theta_i ,$$

and so,

$$(1+x_i)|p_n{}'(x_i)| = \frac{3+\cos\theta_i}{\sin\theta_i}\, n .$$

The function $f(\theta) = (3+\cos\theta)/\sin\theta$ has a unique minimum in the interval $(0, \pi)$ which is assumed at $\theta = \theta_0$, where $\cos\theta_0 = -1/3$, i.e. $\theta_0 \doteq \pi/2 + .340$. Let $i = i_0$ be such that $\pi/2 \leqq \theta_{i_0} < \theta_0$. (The existence of i_0 is trivial if n is odd, and if n is even is assured whenever $n > 4$.) Since $f(\theta_{i_0}) \leqq f(\pi/2) = 3$, we obtain

$$(1+x_{i_0})|p_n{}'(x_{i_0})| \leqq 3n ,$$

and thus, by (2.5) and (7.1),

$$\operatorname{cond}_\infty [V(p_n{}^{(\alpha,\beta)})] > \tfrac{1}{6}(3+\sqrt{8})^n , \qquad (\alpha = \beta = -\tfrac{1}{2}) , \tag{7.2}$$

in agreement with the case $\sigma = 0$ of (6.4).

Consider, next, $\alpha = \beta = \frac{1}{2}$, i.e. $p_n(x) = U_n(2x-1)$. Here we have

$$|p_n(-1)| = U_n(3) = \frac{(3+\sqrt{8})^{n+1}}{2\sqrt{8}}\{1 - (17+6\sqrt{8})^{-n-1}\} , \tag{7.3}$$

and

$$2x_i - 1 = \cos\theta_i , \quad \theta_i = \frac{i}{n+1}\pi , \qquad (i = 1, 2, \cdots, n) .$$

Since now

$$U_n{}'(\cos\theta) = \frac{1}{\sin^3\theta}[\cos\theta \sin(n+1)\theta - (n+1)\sin\theta\cos(n+1)\theta] ,$$

we get

$$p_n{}'(x_i) = 2U_n{}'(\cos\theta_i) = (-1)^{i+1}\frac{2(n+1)}{\sin^2\theta_i} ,$$

and so,

$$(1+x_i)|p_n{}'(x_i)| = \frac{3+\cos\theta_i}{\sin^2\theta_i}(n+1) .$$

In the interval $(0, \pi)$ the function $g(\theta) = (3+\cos\theta)/\sin^2\theta$ takes on its unique minimum at $\theta = \theta_0$, where $\cos\theta_0 = \sqrt{8} - 3$, i.e. $\theta_0 \doteq \pi/2 + .173$. Picking $i = i_0$ such that $\pi/2 \leqq \theta_{i_0} < \theta_0$ (which is always possible if n is odd, and if n is even certainly for $n > 8$), we have $g(\theta_{i_0}) \leqq g(\pi/2) = 3$, and therefore

$$(1 + x_{i_0})|p_n'(x_{i_0})| \leqq 3(n + 1) .$$

Consequently, by (2.5) and (7.3),

$$\begin{aligned} \text{cond}\,[V(p_n^{(\alpha,\beta)})] &\geqq \frac{n}{(6\sqrt{8})(n+1)}\,(3+\sqrt{8})^{n+1}\{1-(17+6\sqrt{8})^{-n-1}\} \\ &\sim \frac{(3+\sqrt{8})^{n+1}}{6\sqrt{8}}\,, \qquad (\alpha = \beta = \tfrac{1}{2},\, n \to \infty)\,, \end{aligned} \tag{7.4}$$

in agreement with the case $\sigma = 1$ of (6.4).

8. For comparison we briefly discuss the case of equidistant abscissas**

$$x_i = i/(n+1)\,, \qquad (i = 1, 2, \cdots, n)\,. \tag{8.1}$$

Here, (2.3) and (2.4) give

$$\text{cond}_\infty\,[V(p_n)] = \frac{n(n+2)(n+3)\cdots(2n+1)}{\min_i \pi_i}\,, \tag{8.2}$$

where

$$\pi_i = (i+n+1) \prod_{j=1;\, j\neq i}^{n} |i-j|\,, \qquad (i = 1, 2, \cdots, n)\,.$$

Observing that

$$\pi_{i+1} = \frac{i+n+2}{i+n+1}\,\frac{i}{n-i}\,\pi_i\,, \qquad (i = 1, 2, \cdots, n-1)\,,$$

and that the function $f(x) = (x+n+2)x/((x+n+1)(n-x))$ is monotonically increasing on the interval $[1, n-1]$, with $f(1) < 1$ (for $n \geqq 3$), $f(n-1) > 1$ (for $n \geqq 2$), $f(n/2) > 1$, $f((n-1)/2) < 1$, it follows that

$$\pi_{i+1} < \pi_i \quad \text{for } i \leqq [(n-1)/2]\,, \qquad \pi_{i+1} > \pi_i \quad \text{for } i > [(n-1)/2]\,.$$

Consequently, the minimum in (8.2) occurs at $i = [(n-1)/2] + 1 = [(n+1)/2]$, and we find that

$$\begin{aligned} \text{cond}_\infty\,[V(p_n)] &= \frac{n^2}{(3n+2)(n+1)}\,\frac{(2n+1)!}{n!(n/2)!^2}\,, \qquad (n \text{ even})\,, \\ \text{cond}_\infty\,[V(p_n)] &= \frac{2n}{3(n+1)}\,\frac{(2n+1)!}{(n+1)!((n-1)/2)!^2}\,, \qquad (n \text{ odd})\,. \end{aligned} \tag{8.3}$$

Therefore, by Stirling's formula,

$$\text{cond}_\infty\,[V(p_n)] \sim \frac{2\sqrt{2}}{3\pi}\,8^n\,, \qquad (n \to \infty)\,. \tag{8.4}$$

Department of Computer Sciences
Purdue University
Lafayette, Indiana 47907

** Consideration of this case was suggested to the author by Professor C. H. Wilcox during a recent conversation.

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An Extension of Szasz's Theorem and Its Application

ELIAS MASRY

Abstract—A classical result in signal theory is the completeness of the exponentials $\{e^{-\mu_n x}\}$ in L_2, the so-called Szasz's theorem. This paper generalizes Szasz's theorem by constructing broad classes of functions $g(x)$ such that the set of functions $\{g(\mu_n x)\}$ is complete in L_2. Application to the problem of alias-free sampling of stochastic processes is considered.

I. Introduction

One of the classical results in signal theory is the completeness of the exponentials $\{e^{-\mu_n x}\}$ in L_2, the so-called Szasz's theorem. It has found numerous applications in systems and signal analysis, control theory, and time-domain approximation of network functions, to mention a few. Let us note that if we let $g(x) = e^{-x}$, then the exponential function $e^{-\mu_n x}$ can be written as $g(\mu_n x)$ and hence, for a proper sequence $\{\mu_n\}$ of numbers, a *single* function $g(x) = e^{-x}$ generates a basis by time scaling.

In this paper we consider an extension of Szasz's theorem to functions other than the exponential with the same unique property of generating a basis. Applications to the problem of alias-free sampling are considered.

Specifically, let L_2^+ be the Hilbert space of all square-integrable Lebesgue-measurable functions over $[0,\infty)$. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of distinct complex numbers. Szasz's theorem [1] states that the set of functions $g_n(x) = e^{-\mu_n x}$, $n = 1,2,\cdots$ with $\operatorname{Re} \mu_n > 0$ is complete in L_2^+ if and only if

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}(\mu_n)}{1 + |\mu_n - \frac{1}{2}|^2} = \infty. \tag{1}$$

In particular, if $\mu_n = n$ or $\mu_n = 1/n$, the sets of functions $\{e^{-nx}\}_{n=1}^{\infty}$ and $\{e^{-(1/n)x}\}_{n=1}^{\infty}$ are complete in L_2^+. We consider an extension of Szasz's theorem in the following sense. Let $g(x) \in L_2^+$ and define

$$g_n(x) = g(\mu_n x), \qquad n = 1,2,\cdots \tag{2}$$

where $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of distinct complex numbers. Under what conditions on $g(x)$ and on $\{\mu_n\}$ is the set of functions $\{g_n(x)\}$ defined by (2) complete in L_2^+? This is a very complex problem and there is no general solution to it. In Section II we solve this problem for two broad classes of functions $g \in L_2^+$. These results are represented by Theorems 2 and 3 of Section II and constitute the main contribution of this paper. In Section III we apply these results to the problem of alias-free sampling of rando processes.

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The author is with the Department of Applied Physics and Information Science, University of California at San Diego, La Jolla, Calif. 92037.

II. An Extension of Szasz's Theorem

Let H be a Hilbert space and $\{f_n(t)\}$ a complete set in
Let A be a bounded linear transformation from H into
Define

$$g_n(t) = (Af_n)(t).$$

We then have the following basic result, the proof of which is given in the Appendix.

Lemma: The set $\{g_n(t)\}$ is complete in H if and only the range $\mathscr{R}(A)$ of the transformation A is dense in H.

Consider next the space L_2^+ of square-integrab Lebesgue-measurable functions defined over $[0,\infty)$. Let be a fixed positive integer greater than one and define every $f \in L_2^+$ the transformation

$$(Tf)(x) = f(rx).$$

Then T is a bounded linear transformation from L_2^+ in L_2^+ with

$$\|Tf\|^2 = (1/r)\|f\|^2, \qquad \forall f \in L_2^+.$$

Hence

$$\|T\| = 1/\sqrt{r}.$$

It is not difficult to see that the operator T is norm Furthermore it will be shown in Theorem 1 that the inver operator T^{-1} exists and is bounded.

Let $\mathscr{P}$ be the class of nonzero functions $g \in L_2^+$ of form

$$g(x) = \sum_{k=-N}^{N} a_k e^{-r^k x},$$

where N is a finite positive integer and the a_k's are arbitra constants. Define a polynomial in z and z^{-1} by

$$P(z) = \sum_{k=-N}^{N} a_k z^k$$

and let

$$P(T) = \sum_{k=-N}^{N} a_k T^k.$$

Then $P(T)$ is a bounded linear transformation mapping to $g(x)$. Furthermore

$$g_n(x) \triangleq g(\mu_n x) = P(T)e^{-\mu_n x}.$$

It then follows by the previous Lemma that the set of fu tions $\{g_n(x)\}$ defined by (9) is complete in L_2^+ if and onl the range $\mathscr{R}(P(T))$ of the operator $P(T)$ is dense in L

Hence the problem reduces to finding the spectrum of the operator $P(T)$. We first prove the following theorem.

Theorem 1: The operator T on L_2^+ has only a continuous spectrum $\sigma_c(T)$ contained in the circle $C_r = \{\lambda: |\lambda| = 1/\sqrt{r}\}$.

Proof: Let $T_\lambda = T - \lambda I$ where I is the identity operator and consider the equation $T_\lambda f = 0$, i.e.,

$$f(rx) = \lambda f(x). \tag{10}$$

It is easily seen by taking norms on both sides of (10) that $f(x) = 0$ a.e. for $|\lambda| \neq 1/\sqrt{r}$. For the case $|\lambda| = 1/\sqrt{r}$, (10) implies

$$\int_t^{rt} |f(x)|^2\, dx = 0, \qquad \forall t \geq 0, \tag{11}$$

which in turn implies $f(x) = 0$ a.e. Hence the operator T_λ has a unique inverse so that the point spectrum is empty. Consider now the norm $\|T_\lambda f\|$

$$\|T_\lambda f\| = \|Tf - \lambda f\| \geq |\, \|Tf\| - |\lambda|\, \|f\|\, |$$

and by (6)

$$\|T_\lambda f\| \geq |(1/\sqrt{r}) - |\lambda|\,|\, \|f\| \tag{12}$$

so that T_λ has a bounded inverse for all λ satisfying $|\lambda| \neq 1/\sqrt{r}$.

Finally consider the range $\mathscr{R}(T_\lambda)$ of the operator T_λ and suppose $f \in [\mathscr{R}(T_\lambda)]^\perp$ where $[\mathscr{R}(T_\lambda)]^\perp$ is the orthogonal complement of the closure of $\mathscr{R}(T_\lambda)$ in L_2^+. Then $f \in N(T_\lambda^*)$, the null space of the adjoint operator T_λ^*. Since T_λ is also normal as can be easily seen, we have $f \in N(T_\lambda)$ so that f belongs to the point spectrum $\sigma_p(T)$. Since $\sigma_p(T)$ is empty, we have $f = 0$ a.e. It then follows that $[\mathscr{R}(T_\lambda)] = L_2^+$ so that $\mathscr{R}(T_\lambda)$ is dense in L_2^+ for all λ.

We thus conclude that all λ satisfying $\lambda \neq 1/\sqrt{r}$ belong to the resolvent set $\rho(T)$ and the circle $C_r = \{\lambda: |\lambda| = 1/\sqrt{r}\}$ contains the continuous spectrum $\sigma_c(T)$.

We now turn to our first basic result.

Theorem 2: Let g be an arbitrary function in $\mathscr{P}$ and $\{\mu_n\}$ a sequence of distinct complex numbers with $\operatorname{Re} \mu_n > 0$. Then the set of functions

$$g_n(x) \triangleq g(\mu_n x), \qquad n = 1,2,\cdots \tag{13}$$

is complete in L_2^+ for every $g \in \mathscr{P}$ if and only if

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re} \mu_n}{1 + |\mu_n - \frac{1}{2}|^2} = \infty. \tag{14}$$

Proof: a) Suppose (14) is not true and let $g(x) = e^{-x} \in \mathscr{P}$. Then the set of functions $\{g_n(x)\}_{n=1}^{\infty}$ is not complete in L_2^+.

b) Suppose (14) is satisfied. Write

$$P(z) = z^{-N} \sum_{k=0}^{2N} a_{k-N} z^k$$

and let $\{\lambda_i\}_{i=1}^{2N}$ be the roots of the polynomial $z^N P(z)$, i.e.,

$$P(z) = z^{-N} \prod_{i=1}^{2N} (z - \lambda_i).$$

Since the operators T_{λ_i} commute with each other, $P(T)$ can be written as

$$P(T) = T^{-N} \prod_{i=1}^{2N} (T - \lambda_i I). \tag{15}$$

By Theorem 1, the range of $T - \lambda_i I$ is dense in L_2^+ for all λ_i. Hence the range

$$\mathscr{R}\left[\prod_{i=1}^{2N} (T - \lambda_i I)\right]$$

is dense in L_2^+. Moreover, the range of T^{-1} is obviously dense in L_2^+. Hence the range of $P(T)$ is dense in L_2^+. It then follows by hypothesis and the previous Lemma that $\{g_n(x)\}_{n=1}^{\infty}$ is complete in L_2^+.

Corollary: The sets of functions $\{g(nx)\}_{n=1}^{\infty}$ and $\{g(x/n)\}_{n=1}^{\infty}$ with $g \in \mathscr{P}$ are complete in L_2^+.

Remark: The idea for the class $\mathscr{P}$ comes from a paper by Neuwirth *et al.* [2]. It should be noted, however, that in [2] the domain of all functions is the compact interval $[0,2\pi]$ and, consequently, the spectrum of the operator T is completely different from ours.

In the next section we will use Theorem 2 with the additional requirement that $g \in \mathscr{P}$ be nonnegative. Since $g(x)$ given by (7) is a mixture of exponentials, we can use previously known sufficient conditions for a mixture of exponentials to be nonnegative [3]. Clearly if all the a_k are nonnegative then $g(x)$ is nonnegative. A nontrivial sufficient condition for $g(x)$ to be nonnegative is given by [3].

$$\sum_{r=-N}^{k} a_r \geq 0, \qquad k = -N,\cdots,0,\cdots,N. \tag{16}$$

As an illustration to Theorem 2 and the discussion following it, we present a special case of the Erlang distribution [4] with density

$$g(x) = \sum_{k=0}^{N} a_k e^{-r^k x}, \tag{17}$$

where

$$a_k = r^k \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{r^j}{r^j - r^k}, \qquad k = 0,1,\cdots,N. \tag{18}$$

Note that the signs of the a_k alternate. It then follows by Theorem 2 that the set of functions $\{g_n(x)\}$ defined by (13) with $g(x)$ given by (17) is complete in L_2^+.

We now extend the results of Theorem 2 to a larger class of functions g. We recall first that the resolvent transformation $R(\lambda,T) = (T - \lambda I)^{-1}$ exists and is bounded for all complex-valued $\lambda \in \rho(T)$. Let $\mathscr{A}$ be the class of complex-valued functions $a(\lambda)$ that are analytic in some neighborhood D_a of the circle $C_r = \{\lambda: |\lambda| = 1/\sqrt{r}\}$. We define the operator $a(T)$ by [5]

$$a(T) = \frac{1}{2\pi i} \int_B a(\lambda) R(\lambda; T)\, d\lambda. \tag{19}$$

B consists of a finite number of rectifiable Jordan curves oriented in the positive sense and is the boundary of an open set 0 containing the circle C_r such that $0 \cup B$ is contained in D_a. Then the operator $a(T)$ on L_2^+ is bounded, linear, and uniquely defined [5].

Define the class $\mathscr{F}$ of functions $g \in L_2^+$ by

$$\mathscr{F} = \{g \in L_2^+ : g = a(T)e^{-x},\ a(\lambda) \in \mathscr{A}\} \tag{20}$$

and note that, in particular, $\mathscr{P} \subset \mathscr{F}$. We now state Theorem 3.

Theorem 3: Let $a(\lambda)$ be an arbitrary function in $\mathscr{A}$ and let $g = a(T)e^{-x} \in \mathscr{F}$. Let $\{\mu_n\}$ be a sequence of distinct complex numbers with $\operatorname{Re} \mu_n > 0$. If $a(\lambda)$ does not vanish for any $\lambda \in C_r$ then the set of functions

$$g_n(x) = g(\mu_n x), \qquad n = 1,2,\cdots \tag{21}$$

is complete in L_2^+ for every $g \in \mathscr{F}$ if and only if

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re} \mu_n}{1 + |\mu_n - \frac{1}{2}|^2} = \infty. \tag{22}$$

Proof: a) The necessity is trivial if we let $a(\lambda) = 1$.

b) Let B_1 and B_2 be the circles $B_1 = \{\lambda : |\lambda| = (1/\sqrt{r}) + \varepsilon_1\}$ and $B_2 = \{\lambda : |\lambda| = (1/\sqrt{r}) - \varepsilon_2\}$, where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. For sufficiently small ε_1 and ε_2, B_1 and B_2 are in the domain of analyticity of $a(\lambda)$. We then have

$$a(T) = \frac{1}{2\pi i}\int_{B_1} a(\lambda)R(\lambda,T)\,d\lambda + \frac{1}{2\pi i}\int_{B_2} a(\lambda)R(\lambda,T)\,d\lambda. \tag{23}$$

Now on B_1, $|\lambda| > \|T\|$ so that [6]

$$R(\lambda,T) = \sum_{n=1}^{\infty} \lambda^{-n}T^{n-1}$$

and on B_2, $|\lambda| < 1/\|R(0,T)\| = \|T\|$ so that [6]

$$R(\lambda,T) = \sum_{n=0}^{\infty} \lambda^n[R(0,T)]^{n+1} = \sum_{n=0}^{\infty} \lambda^n T^{-(n+1)}.$$

It then follows by (23) that

$$a(T) = \sum_{n=1}^{\infty} c_{n-1}T^{n-1} + \sum_{n=0}^{\infty} c_{-n-1}T^{-n-1}$$

or

$$a(T) = \sum_{n=-\infty}^{\infty} c_n T^n, \tag{24}$$

where c_n is the nth coefficient of the expansion of $a(\lambda)$ in a Laurant series

$$a(\lambda) = \sum_{n=-\infty}^{\infty} c_n \lambda^n$$

valid in an annulus $(1/\sqrt{r}) - \varepsilon_2 < |\lambda| < (1/\sqrt{r}) + \varepsilon_1$. It then follows by (24) that

$$g(x) = \sum_{k=-\infty}^{\infty} c_k e^{-r^k x} \tag{25}$$

so that

$$g_n(x) = a(T)e^{-\mu_n x}, \qquad n = 1,2,\cdots. \tag{26}$$

By the Lemma, the set of functions $\{g_n(x)\}$ is complete in L_2^+ if and only if the range of the operator $a(T)$ is dense in L_2^+. Now by the spectral mapping theorem [5], the spectrum $\sigma(a(T))$ of $a(T)$ is given by

$$\sigma(a(T)) = a(\sigma(T))$$

so that

$$\sigma(a(T)) = \{a(\lambda);\ \lambda \in \sigma(T)\} \subset \{a(\lambda);\ \lambda \in C_r\}. \tag{27}$$

Since by assumption $a(\lambda)$ does not vanish for any $\lambda \in C_r$, we have that $0 \notin \sigma(a(T))$. Consequently, $a(T)$ has a bounded inverse and hence the range of $a(T)$ is dense in L_2^+.

Corollary: Let $a(\lambda) \in \mathscr{A}$ not vanish for $\lambda \in C_r$. If $g = a(T)e^{-x}$, then the sets of functions $\{g(nx)\}_{n=1}^{\infty}$ and $\{g(x/n)\}_{n=1}^{\infty}$ are complete in L_2^+.

As an example to Theorem 3, we note that any series

$$a(\lambda) = \sum_{k=-\infty}^{\infty} a_k \lambda^k$$

converging in an annulus $\alpha < |\lambda| < \beta$ such that $0 \le \alpha < (1/\sqrt{r}) < \beta < \infty$ generates a function $g \in \mathscr{F}$ given by

$$g(x) = \sum_{k=-\infty}^{\infty} a_k e^{-r^k x}.$$

Moreover, if the Fourier series

$$a_1(e^{ix}) = \sum_{k=-\infty}^{\infty} (a_k/r^{k/2})e^{ikx} \tag{28}$$

does not vanish for $0 \le x \le 2\pi$ then the set of functions $\{g_n(x) = g(\mu_n x)\}$ with $\{\mu_n\}$ satisfying (22) is complete in L_2^+. Note that $g(x)$ is also in L_1^+ since $a(\lambda)$ converges uniformly and absolutely for $\lambda \in C_r$. Hence if the a_k are nonnegative, $g(x)$ can be normalized to become a probability density function.

III. Applications to Random Sampling

Let $x(t)$ be a real second-order mean-square-continuous weakly stationary stochastic process with zero mean and spectral distribution $S(\lambda)$. The process $x(t)$ is sampled at times $\{t_n\}$ where $\{t_n\}$ is a stationary point process independent of $x(t)$. It is required to perfectly reconstruct $S(\lambda)$ from the correlation sequence $\{c(n)\}$ of the discrete-parameter weakly stationary process $\{x(t_n)\}$, i.e., from

$$c(n) = E[x(t_{m+n})x(t_m)], \qquad n = 0,\pm 1,\cdots, \tag{29}$$

where the expectation is taken over both $x(t)$ and the point process $\{t_n\}$. It is assumed that the point process $\{t_n\}$ has a finite average number of points β per unit time and that the distribution function $F_n(\tau)$ of $t_{m+n} - t_m$ does not depend on m. A detailed discussion of the problem can be found in [7] and [8]. We note here that by taking expectations in (29) first with respect to $x(t)$ and then with respect to $\{t_n\}$ we obtain

$$c(\pm n) = \int_0^{\infty} C(\tau)\,dF_n(\tau), \qquad n = 1,2,\cdots, \tag{30}$$

where $C(\tau)$ is the covariance function of $x(t)$. Suppose now that $F_n(t)$ is absolutely continuous with corresponding density function $f_n(t) \in L_1{}^+ \cap L_2{}^+$ and suppose that $C(\tau) \in L_2$. Then (30) becomes

$$c(\pm n) = \int_0^\infty C(\tau) f_n(\tau)\, d\tau, \qquad n = 1,2,\cdots. \tag{31}$$

It then follows by (31) that if $\mathscr{S}_2$ denotes the family of absolutely continuous spectral distributions $S(\lambda)$ with corresponding spectral density $s(\lambda) \in L_1 \cap L_2$, the covariance function $C(\tau)$ can be uniquely recovered from $\{c(n)\}$ if and only if the set of functions $\{f_n(t)\}_{n=1}^\infty$ is complete in $L_2{}^+$. In such a case we say that the sampling sequence $\{t_n\}$ is alias-free relative to $\mathscr{S}_2$.

Various classes of alias-free point processes were constructed in [8]. We limit ourselves in this paper to the class of simply additive random sampling, which appears to be the natural counterpart to periodic sampling. Let

$$t_n = t_{n-1} + \gamma, \qquad n = 0, \pm 1, \cdots, \tag{32}$$

where γ is a fixed random variable with density function $f(x)$ over $[0,\infty)$ and a finite mean $1/\beta$. It is then apparent that the sampling instants $\{t_n\}$ are equally spaced with probability one. From (32) we conclude that the probability density function of $t_{m+n} - t_m$ is independent of m and is given by

$$f_n(t) = (1/n) f(t/n), \qquad n = 1,2,\cdots. \tag{33}$$

We thus have that simply additive random sampling is alias-free relative to $\mathscr{S}_2$ if and only if the set of density functions $\{f_n(t)\}$ given by (33) is complete in $L_2{}^+$.

In [8] we concluded by Szasz's theorem that simply additive random sampling with exponential distribution is alias-free relative to the family $\mathscr{S}_2$ of spectral distributions. As a consequence of Theorems 2 and 3 of Section II, we can generalize this result. Let $\mathscr{F}_1 \subset \mathscr{F}$ be the class of density functions of the form

$$g(x) = \sum_{k=-\infty}^{\infty} a_k e^{-r^k x}$$

such that the Fourier series

$$a_1(e^{ix}) = \sum_{k=-\infty}^{\infty} a_k \frac{e^{ikx}}{(\sqrt{r})^k}$$

does not vanish for $0 \le x \le 2\pi$. Members of $\mathscr{F}_1$ were shown to exist in abundance. Note that the assumption on $a_1(e^{ix})$ can be dropped if $g(x)$ is given by a finite series (cf. Theorem 2). We have by Theorem 3 the following result.

Theorem 4: Simply additive random sampling generated by a random variable γ with probability density $f \in \mathscr{F}_1$ is alias-free relative to the family $\mathscr{S}_2$ of spectral distributions.

The special Erlang density (17) is an example for which Theorem 4 is applicable.

Remark: The reconstruction of $C(\tau) \in L_2$ from $\{c(n)\}$ is very simple when $\{t_n\}$ is alias free. We orthonormalize the complete set of functions $\{f_n(t)\}_{n=1}^\infty$ to obtain $\{\phi_n(t)\}_{n=1}^\infty$, i.e.,

$$\phi_n(t) = \sum_{k=1}^{n} d_{k,n} f_k(t), \qquad n = 1,2,\cdots, \tag{34}$$

where the coefficients $\{d_{k,n}\}$ are obtained by the Gram–Schmidt procedure. We then have

$$C(\tau) = \sum_{n=1}^{\infty} a(n)\phi_n(t) \tag{35}$$

in L_2, where

$$a(n) = \sum_{k=1}^{n} d_{k,n} c(k), \qquad n = 1,2,\cdots. \tag{36}$$

IV. Appendix

Proof of the Lemma

a) Suppose $\{g_n(t)\}$ is complete in H. Then every $f \in H$ can be approximated by a linear combination of the g_n such that

$$\left\| f - \sum_{k=1}^{N} c_{k,N} g_k \right\| < \varepsilon. \tag{A1}$$

Define

$$h = \sum_{k=1}^{N} c_{k,N} f_k. \tag{A2}$$

We then have $Ah \in \mathscr{R}(A)$ and

$$\|f - Ah\| < \varepsilon. \tag{A3}$$

Hence $\mathscr{R}(A)$ is dense in H.

b) Suppose $\mathscr{R}(A)$ is dense in H. Then if $f \in H$ and

$$(f, Ah) = 0, \qquad \forall h \in H \tag{A4}$$

we have $f = 0$ a.e. Since $\{f_n\}$ spans H, we have by (A4)

$$(f, Af_n) = 0, \qquad \forall \text{ integer } n \Rightarrow f = 0 \text{ a.e.}$$

or

$$(f, g_n) = 0, \qquad \forall \text{ integer } n \Rightarrow f = 0 \text{ a.e.} \tag{A5}$$

and hence $\{g_n\}$ is complete in H.

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numerical inversion of the Laplace transform

some aspects of Gaussian quadrature formula for the

R. Piessens

Institute for Applied Mathematics, University of Leuven, Celestijnenlaan 200 B, B-3030, Heverlee, Belgium

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The optimal addition of abscissas to Gaussian quadrature formulae for the numerical evaluation of the Bromwich integral is discussed and Gaussian quadrature rules with a preassigned abscissa at infinity are studied. Techniques are given for the efficient calculation of the abscissas of the unconstrained Gauss formulae.

GAUSSIAN QUADRATURE

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Introduction

Let $F(p)$ be a given Laplace transform and $f(t)$ the corresponding original function. A very simple numerical method for the inversion of the Laplace transform is the numerical integration of the Bromwich inversion formula

$$f(t) = \frac{1}{2\pi j}\int_L e^{pt} F(p)\, dp \tag{1}$$

where L is defined as the line $\{p: Re(p) = c\}$ in the complex plane, and where c is chosen so that L lies to the right of all singularities of $F(p)$, but is otherwise arbitrary.

Substituting

$$pt = u$$

and

$$F(u/t) = u^{-s}\, G(u)$$

where s is a parameter, (1) yields

$$f(t) = \frac{1}{2\pi jt}\int_{L'} e^{u}\, u^{-s}\, G(u)\, du \tag{2}$$

where L' is the line $\{u: Re(u) = tc\}$.

We consider now an approximate formula for the evaluation of the integrals in (2)

$$\frac{1}{2\pi j}\int_{L'} e^{u}\, u^{-s}\, G(u)\, du \simeq \sum_{k=1}^{N} A_k^{(s)}\, G(u_k^{(s)}) \tag{3}$$

Piessens (1969a) has chosen the abscissas $u_k^{(s)}$ as equidistant real numbers. The weights $A_k^{(s)}$ are then determined such that formula (3) is exact whenever $G(u)$ is an arbitrary polynomial in u^{-1}, of degree $\leqslant N-1$. Formula (3) is then an integration formula of interpolatory type.

Krylov and Skoblya (1961 and 1969), Luke (1969), Piessens (1969b) and Salzer (1955 and 1961) have given formulas for the abscissas $u_k^{(s)}$ and the weights $A_k^{(s)}$ such that (3) is exact whenever $G(u)$ is a polynomial in u^{-1}, of degree $\leqslant 2N-1$, in other words, such that (3) has a precision degree $2N-1$. In this sense, (3) is a N-point Gaussian quadrature formula, and we shall refer to it by the symbol G_N.

The abscissas of the G_N-formula are the zeros of the polynomial in p^{-1}

$$P_{N,s}(p^{-1}) = (-1)^N\, {}_2F_0(-N, N+s-1; p^{-1}) \tag{4}$$

The weights are given by

$$A_k^{(s)} = (-1)^{N-1}\frac{(N-1)!}{N\,\Gamma(N+s-1)}\, u_k^{-2}\left[\frac{2N+s-2}{P_{N-1,s}(1/u_k)}\right]^2 \tag{5}$$

where $u_k = u_k^{(s)}$.

Substituting (3) in (2), we obtain

$$f(t) \simeq t^{s-1}\sum_{k=1}^{N} A_k^{(s)}\left(\frac{u_k^{(s)}}{t}\right)^s F\left(\frac{u_k^{(s)}}{t}\right) \tag{6}$$

The Gaussian formulas are much more accurate than the interpolating formulas, but they have several shortcomings. Firstly, no convenient rule exists for determining at the outset the order N such that the desired accuracy is obtained. The practical procedure is then the use of a series of Gaussian formulas with increasing order. If agreement occurs of two successive approximations to within the desired accuracy, the last computed value is retained as definitive result. This procedure has the well-known disadvantage of using different values of the abscissas for different values of the order.

A second disadvantage is that, when N is large, the weights are also large. Since this leads to considerable cancellation errors, the use of a sequence of G_N-formulas with increasing N is not recommendable.

Piessens (1969c) has proposed the use of the Gaussian quadrature formulas in combination with new integration formulas obtained by optimal addition of abscissas to Gaussian quadrature formulas. He has given the formulas only for the case $s = 1$. Here, in Section 1, we shall generalise his results for arbitrary s.

Another disadvantage of formula (6) is that for each value of t, the Laplace transform $F(p)$ must be calculated N times. In Section 2, we shall consider Gaussian quadrature formulas with a preassigned abscissa at infinity. The inversion-formula is then

$$f(t) \simeq W_0^{(s)}\, L t^{s-1} + t^{-1}\sum_{k=1}^{N} W_k^{(s)}\,(v_k^{(s)})^s\, F(v_k^{(s)}/t) \tag{7}$$

where $v_k^{(s)}$ and $W_k^{(s)}$, $k = 0, 1, 2, \ldots, N$ are the abscissas and weights, and

$$L = \lim_{p\to\infty} p^s F(p) \tag{8}$$

Formula (7) has a precision degree $2N$, and the computation time for the evaluation of (7) is approximately the same as for the evaluation of G_N-formula, which has a precision degree $2N-1$.

1. Optimal addition of abscissas to the Gaussian quadrature formula

The purpose of this Section is the calculation of abscissas and weights of the formula

$$\frac{1}{2\pi j}\int_{L'} e^{u}\, u^{-s}\, G(u)\, du \simeq \sum_{k=1}^{N} B_k^{(s)}\, G(u_k^{(s)}) + \sum_{k=1}^{M} C_k^{(s)}\, G(w_k^{(s)}) \tag{9}$$

where $u_k^{(s)}$, $k = 1, 2, \ldots, N$ are the abscissas of the G_N-rule.

s, we know that, if there exists a polynomial $Q_{M,s}(p^{-1})$ M, with the property

$$\int_L e^p p^{-s} P_{N,s}(p^{-1}) Q_{M,s}(p^{-1}) p^{-r} dp = 0 \quad (10)$$

, 1, . . ., $M-1$, we can determine the weights $B_k^{(s)}$ and such that the precision degree of (9) is $N + 2M - 1$, if abscissas $w_k^{(s)}$ are the zeros of $Q_{M,s}(p^{-1})$. This degree of ision is maximal. To find the polynomial $Q_{M,s}(p^{-1})$, we alculate in the first place the moments

$$M_{N,r} = \frac{1}{2\pi j}\int_{L'} e^p p^{-s} P_{N,s}(p^{-1}) p^{-r} dp \quad (11)$$

t is obvious that

$$M_{N,r} = 0 \quad \text{for } r = 0, 1, 2, \ldots, N-1.$$

Further, we have

$$M_{N,N+k} = (-1)^N \frac{(N+k)!}{\Gamma(2N+k+s)\,k!} \quad (12)$$

or $k = 0, 1, 2, \ldots$

To demonstrate (12), we note that

$$M_{N,r} = \Phi_r(1)$$

vhere

$$\Phi_r(t) = \mathscr{L}^{-1}\{p^{-(s+r)} P_{N,s}(p^{-1})\}$$

vhere $\mathscr{L}^{-1}$ denotes the inverse Laplace transform.

From

$$\mathscr{L}^{-1}\{p^{-a} P_{N,s}(p^{-1})\} = (-1)^N \frac{t^{a-1}}{\Gamma(a)}\, {}_2F_1(-N, N+s-1; a; t)$$

vhere a is a positive real number, we obtain

$$M_{N,r} = \frac{(-1)^N}{\Gamma(r+s)}\, {}_2F_1(-N, N+s-1; r+s; 1) \quad (13)$$

tituting the relation

$$F_1(-N, N+s-1; r+s; 1) = \frac{\Gamma(r+s)\,\Gamma(r+1)}{\Gamma(r+s+N)\,\Gamma(r+1-N)}$$

1 (13), we have immediately the required formula (12).

Further, let us set

$$Q_{M,s}(p^{-1}) = p^{-M} + a_{M-1}p^{-M+1} + \ldots + a_1 p^{-1} + a_0 \quad (14)$$

ubstituting (14) in (10), we see that

(i) If $M \leqslant N/2$, the condition (10) is satisfied for an arbitrary polynomial $Q_{M,s}(p^{-1})$. Since the corresponding weights are then zero, this case is not to consider.

(ii) If $N/2 < M \leqslant N$, the polynomial $Q_{M,s}(p^{-1})$ which satisfies (10) does not exist.

(iii) If $M = N + 1$, the required polynomial $Q_{M,s}(p^{-1})$ exists and is unique.

'e restrict ourselves to case (iii), which is the most important r practical applications.

The coefficients $a_N, a_{N-1}, \ldots, a_0$ are calculated recursively

$$a_N = -\frac{M_{N,N+1}}{M_{N,N}}$$

$$a_{N-k} = -\frac{M_{N,N+k+1} + M_{N,N+k}a_N + \ldots + M_{N,N+1}a_{N-k+1}}{M_{N,N}} \quad (15)$$

r $k = 1, 2, \ldots, N$.

Jsing (12) the relations (15) become

$$a_N = -\frac{N+1}{2N+s}$$

$$a_{N-k} = -[E_{N,k+1} + E_{N,k}\,a_N + \ldots + E_{N,1}\,a_{N-k+1}]$$

r $k = 1, 2, \ldots, N$, where

$$\frac{(N+1)(N+2)\ldots(N+i)}{(2N+s)(2N+s+1)\ldots(2N+s+i-1)\,i!} \quad (16)$$

The required additional abscissas $w_k^{(s)}$ of the new quadrature formula (9) are the roots of

$$Q_{N+1,s}(p^{-1}) = 0$$

There is a certain regularity in the distribution of the zeros of $Q_{N+1,s}$, with respect to the zeros of $P_{N,s}$. This regularity is helpful for the determination of $w_k^{(s)}$ with the aid of an iteration formula. For $s = 1$, the position of the abscissas $u_k^{(s)}$ and $w_k^{(s)}$ is shown graphically by Piessens (1969c).

The formulae for the weights are

$$B_k^{(s)} = \frac{1}{2\pi j\, P'_{N,s}(u_k^{-1})\, Q_{N+1,s}(u_k^{-1})}\int_{L'} e^p p^{-s} \frac{P_{N,s}(p^{-1})\, Q_{N+1,s}(p^{-1})}{p^{-1} - u_k^{-1}} dp \quad (17)$$

for $k = 1, 2, \ldots, N$, and

$$C_k^{(s)} = \frac{1}{2\pi j\, P_{N,s}(w_k^{-1})\, Q'_{N+1,s}(w_k^{-1})}\int_{L'} e^p p^{-s} \frac{P_{N,s}(p^{-1})\, Q_{N+1,s}(p^{-1})}{p^{-1} - w_k^{-1}} dp \quad (18)$$

for $k = 1, 2, \ldots, N+1$, and where $w_k = w_k^{(s)}$ and $u_k = u_k^{(s)}$.

In (17) and (18) $P'_{N,s}$ and $Q'_{N+1,s}$ are the derivatives with respect to $1/p$.

Using the equality

$$P'_{N,s}(u_k^{-1}) = \frac{N}{2N+s-2} u_k^2 P_{N-1,s}(u_k^{-1}) \quad (19)$$

and applying the orthogonality property of the polynomials $P_{N,s}(p^{-1})$, (17) becomes

$$B_k^{(s)} = \frac{(-1)^N (N-1)!\,(2N+s-2)}{\Gamma(2N+s)\, u_k^2 P_{N-1,s}(u_k^{-1})\, Q_{N+1,s}(u_k^{-1})} + A_k^{(s)} \quad (20)$$

where $A_k^{(s)}$ is the corresponding weight of the G_N-formula, given by (5).

In the same way, we obtain

$$C_k^{(s)} = \frac{(-1)^N N!}{\Gamma(2N+s)\, P_{N,s}(w_k^{-1})\, Q'_{N+1,s}(w_k^{-1})} \quad (21)$$

For a table of abscissas and weights of this quadrature formula, for the case $s = 1/2$, see Piessens (1970).

In order to compare the weights of the G_N-formula and of the formula with optimally added abscissas, we give in **Table 1** the largest modulus of the weights for both formulas ($s = 1$).

Table 1 Comparison of the weights of both quadrature formulae

N	GAUSSIAN FORMULA G_N		$(2N+1)$-POINT FORMULA WITH OPTIMALLY ADDED ABSCISSAS	
	PRECISION DEGREE	MAX|WEIGHT|	PRECISION DEGREE	MAX|WEIGHT|
6	11	$1{\cdot}2 \times 10^2$	19	$9{\cdot}4 \times 10^1$
8	15	$1{\cdot}3 \times 10^3$	25	$1{\cdot}2 \times 10^3$
10	19	$1{\cdot}5 \times 10^4$	31	$1{\cdot}8 \times 10^4$
12	23	$1{\cdot}9 \times 10^5$		
16	31	$1{\cdot}9 \times 10^7$		

In this example we illustrate the difference in loss of significance, if we use the G_N-formula or the new formula with the same accuracy.

We carry out the inversion of

$$F(p) = \frac{1}{\sqrt{p^2 + 1}}$$

in single precision on the IBM 360/44, with the G_{10}-formula and with the new 17-point formula. Results are given in Table 2. In both cases, for small values of t ($t \leqslant 12$) the errors are due to loss of significant figures, but they are considerably smaller in the second case.

Table 2 Numerical results for the example 1

t	EXACT ORIGINAL FUNCTION $J_0(t)$	ERRORS = |EXACT VALUE — APPROXIMATE VALUE|	
		GAUSSIAN FORMULA G_{10}	17-POINT FORMULA WITH OPTIMALLY ADDED ABSCISSAS
2·0	0·2239	$4{\cdot}6 \times 10^{-3}$	$1{\cdot}2 \times 10^{-4}$
4·0	−0·3971	$1{\cdot}9 \times 10^{-3}$	$7{\cdot}0 \times 10^{-5}$
6·0	0·1506	$5{\cdot}6 \times 10^{-4}$	$1{\cdot}1 \times 10^{-4}$
8·0	0·1717	$2{\cdot}5 \times 10^{-3}$	$2{\cdot}5 \times 10^{-4}$
10·0	−0·2459	$3{\cdot}6 \times 10^{-3}$	$5{\cdot}1 \times 10^{-5}$
12·0	0·0477	$4{\cdot}0 \times 10^{-5}$	$2{\cdot}5 \times 10^{-4}$
14·0	0·1711	$1{\cdot}0 \times 10^{-2}$	$1{\cdot}5 \times 10^{-4}$
16·0	−0·1749	$1{\cdot}8 \times 10^{-2}$	$9{\cdot}0 \times 10^{-4}$
18·0	−0·0134	$6{\cdot}6 \times 10^{-2}$	$9{\cdot}3 \times 10^{-3}$

2. Gaussian quadrature formulae with a preassigned abscissa at infinity

We consider now the quadrature formula

$$\frac{1}{2\pi j}\int_{L'} e^u u^{-s} G(u)\,du \simeq W_0^{(s)} L^* + \sum_{k=1}^{N} W_k^{(s)} G(v_k^{(s)}) \tag{22}$$

where

$$L^* = \lim_{u\to\infty} G(u)$$

We try to determine the abscissas $v_k^{(s)}$ and the weights $W_k^{(s)}$, such that the precision degree of (22) is $2N$.

A result of the theory of Gaussian quadrature rules with preassigned abscissas is that the abscissas $v_k^{(s)}$, $k = 1, 2, \ldots, N$, are the zeros of a polynomial $Q_{N,s}(p^{-1})$ in p^{-1}, of degree N, which has the property

$$\frac{1}{2\pi j}\int_L e^p p^{-s} p^{-1} Q_{N,s}(p^{-1})\, p^{-r}\,dp = 0 \tag{23}$$

for $r = 0, 1, 2, \ldots, N-1$.

It is evident that

$$Q_{N,s}(p^{-1}) = P_{N,s+1}(p^{-1}) \tag{24}$$

or

$$Q_{N,s}(p^{-1}) = (-1)^N\, {}_2F_0(-N, N+s; p^{-1}) \tag{25}$$

The abscissas of (22) are thus

$$v_k^{(s)} = u_k^{(s+1)}, k = 1, 2, \ldots, N \tag{26}$$

The general formula for the weights is

$$W_k^{(s)} = \frac{1}{2\pi j}\int_L e^p p^{-s} \frac{p^{-1} Q_{N,s}(p^{-1})}{(p^{-1} - q_k)\,[Q_{N,s}(q_k) + q_k Q'_{N,s}(q_k)]}\,dp \tag{27}$$

where

$q_k = 1/v_k^{(s)}$, $k = 1, 2, \ldots, N$

Setting $k = 0$ in (27), we obtain

$$W_0^{(s)} = \frac{1}{2\pi j}\int_L e^p p^{-s} \frac{Q_{N,s}(p^{-1})}{Q_{N,s}(0)}\,dp \tag{28}$$

or

$$W_0^{(s)} = \frac{1}{\Gamma(s)}\, {}_2F_1(-N, N+s; s; 1) = \frac{(-1)^N N!}{\Gamma(s+N)} \tag{29}$$

For $k = 1, 2, \ldots, N$, we obtain

$$W_k^{(s)} = \frac{1}{2\pi j}\int_L e^p p^{-s} \frac{p^{-1} Q_{N,s}(p^{-1})}{(p^{-1} - q_k)q_k Q'_{N,s}(q_k)}\,dp \tag{30}$$

or

$$W_k^{(s)} = u_k^{(s+1)} A_k^{(s+1)} \tag{31}$$

Using the tables given by Skoblya (1964), Piessens (1969b) or Krylov and Skoblya (1969) or using the method described in Section 3, the abscissas $v_k^{(s)}$ and the weights $W_k^{(s)}$ can be calculated easily.

Numerical example 2

In Table 3. the results are given of the inversion of

$$F(p) = (p^2 + 1)^{-\frac{1}{2}}$$

using the formulae (6) and (7) with $s = 1$, $N = 4$. The computation work for both formulae is approximately the same.

Table 3 Numerical results for the example 2

t	EXACT ORIGINAL FUNCTION $J_0(t)$	ERRORS = |EXACT VALUE − APPROXIMATE VALUE|	
		GAUSSIAN FORMULA G_4	FORMULA WITH ABSCISSA AT INFINITY
1·0	0·765197687	$0{\cdot}91 \times 10^{-7}$	$0{\cdot}81 \times 10^{-8}$
2·0	0·223890779	$0{\cdot}23 \times 10^{-4}$	$0{\cdot}44 \times 10^{-5}$
3·0	−0·260051955	$0{\cdot}93 \times 10^{-3}$	$0{\cdot}44 \times 10^{-4}$
4·0	−0·397149810	$0{\cdot}30 \times 10^{-2}$	$0{\cdot}10 \times 10^{-2}$
5·0	−0·177596771	$0{\cdot}18 \times 10^{-1}$	$0{\cdot}92 \times 10^{-2}$
6·0	0·150645257	$0{\cdot}79 \times 10^{-1}$	$0{\cdot}20 \times 10^{-1}$
7·0	0·300079271	$0{\cdot}85 \times 10^{-1}$	$0{\cdot}17 \times 10^{-1}$

3. Techniques for the calculation of Gaussian abscissas for the Bromwich integral

It is proved by Van Rossum (1969) that the zeros of $P_{N,s}(p^{-1})$ lie in the right half-plane of the complex plane, if s is an even integer.

Some computations (Krylov and Skoblya (1969) and Piessens (1969b)) confirm the assumption that this property holds also for other values of s, but no proof is known.

If the order of the formula is odd, there is only one real abscissa; if the order is even, there is no real abscissa. Only the abscissas in the first quadrant of the complex plane and the corresponding weights are calculated. The other abscissas and weights are complex conjugated.

The abscissas $u_k^{(s)}$ of the N-th order formula are the zeros of the polynomial $P_{N,s}(p^{-1})$ given by formula (4). They can be calculated by the Newton-Raphson iteration method or, even more efficiently, by the iteration method of third order

$$u^* = u + (-2 + s + 2N)\,u^2 \left[\frac{V_1}{N}\frac{P_{N,s}(u)}{P_{N-1,s}(u)} + \frac{V_2}{2N^2}\left(\frac{P_{N,s}(u)}{P_{N-1,s}(u)}\right)^2\right] \tag{32}$$

where

and where u is an approximate value for the inverse of the zero of $P_{N,s}(p^{-1})$ and u^* is the improved value.

The polynomial values are calculated using the recurrence relation

$$P_{N,s}(x) = (a_N x + b_N) P_{N-1,s}(x) + c_N P_{N-2,s}(x) \qquad (33)$$

for $N \geqslant 2$, where

$$a_N = \frac{(2N + s - 3)(2N + s - 2)}{(N + s - 2)}$$

$$b_N = \frac{(2N + s - 3)(2 - s)}{(N + s - 2)(2N + s - 4)} \qquad (34)$$

$$c_N = \frac{(2N + s - 2)(N - 1)}{(N + s - 2)(2N + s - 4)}$$

and

$$P_{0,s}(x) = 1$$
$$P_{1,s}(x) = sx - 1 \qquad (35)$$

However, for small values of s, this recurrence formula gives large roundoff errors. The roundoff errors are considerably smaller if the recurrence formula is started at $N = 3$, thus using also the explicit expression

$$P_{2,s}(x) = (s + 1)(s + 2)x^2 - 2(s + 1)x + 1 \qquad (36)$$

The derivative, which is required for the Newton-Raphson method, can easily be calculated using the expression

$$\frac{d}{dp} P_{N,s}(p^{-1}) = -\left(Np^{-1} + \frac{N}{2N + s - 2}\right) P_{N,s}(p^{-1}) - \frac{N}{2N + s - 2} P_{N-1,s}(p^{-1}) \qquad (37)$$

$$u_1^{(s)} \cong 4N/3 + s - 1\cdot 5 \quad \text{for } N \text{ odd} \qquad (38)$$

or

$$u_1^{(s)} \cong (4N/3 + s - 1\cdot 5) + j(1\cdot 6 + 0\cdot 07s) \quad \text{for } N \text{ even} \qquad (39)$$

and further

$$u_{k+1}^{(s)} \cong (u_k^{(s)} + 0\cdot 67\, N) \exp(j\phi_k) - 0\cdot 67\, N \qquad (40)$$

for $k = 1, 2, \ldots, N - 1$, where

$$\phi_k = 0\cdot 034(2N + 30)/(N - 1) \qquad (41)$$

for $k = 1, 2, \ldots, N - 2$, and

$$\phi_{N-1} = 1\cdot 5\, \phi_{N-2} \qquad (42)$$

The formulae (36)-(41) were found experimentally and are based on a certain regularity in the distribution of the abscissas in the complex plane (see Piessens, 1970). Indeed, for fixed N and s, the zeros lie very nearly on a circle with centre on the negative real axis. The radius of this circle is approximately an increasing linear function of N and s. For fixed N and s, the angular distance between two consecutive zeros is nearly constant.

The starting values (38)-(40) are tested for $s = 0\cdot 1(0\cdot 1)4\cdot 0$ and $N = 4(1)12$, using the Newton-Raphson method, and for $s = 0\cdot 1(0\cdot 1)6\cdot 0$ and $N = 8(1)12$, using the iteration formula (32). Each abscissa was found to at least 10 accurate significant figures, in at most six steps of the Newton-Raphson method and in at most four steps of the iteration method based on (32).

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Book review

Mathematical Model Building in Economics and Industry (Second Series), by M. G. Kendall (editor), 1970; 277 pages. (*Charles Griffin & Co.*, £3·75)

This is the second volume of essays by various leading authorities on topics connected with Econometric model building. All the papers are written from the practical point of view by people active in the fields of actual applications, so that their papers tend to be factual rather than theoretical. These essays are all of a very high standard, as was volume one, but special mention might be made of Professor Ball's two papers which open and close the book. They both give clear illustrative examples of the ideas under discussion and outlines of several of the most important techniques in this field.

Other interesting papers are one by Professor Pyatt, on various ways of estimating brand loyalties among the consumers, and by Dr. Hughes, on a computable model for assessment of the effects of advertising media. All the papers have actual numerical examples in their text, and lists of further references, at the end. Another paper surveying a wide field of application, is that by Dr. Orcutt on Microanalytic models, and the various methods used for solving them. He gives a very good outline of the whole field and discusses the wide range of computing methods used in this field. Again a valuable list of references is given.

Mr. Duffett and Mr. Chadwick give two papers on manpower planning and staff control which again emphasise the practical approach and outline the actual computing techniques which can be used. This book can be highly recommended to every student of modern econometrics and to those who have to apply modern computers in this field.

L. J. Slater (Cambridge)

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An Efficient Method of Numerical Inversion of Laplace Transforms[1]

By

O. Wing, New York

Laguerre

(Received April 26, 1967)

Summary. The COOLEY-TUKEY algorithm for the calculation of complex FOURIER Series is applied to the numerical inversion of Laplace Transforms in which the original function is expanded into Laguerre polynomials.

Zusammenfassung. Der COOLEY-TUKEY-Algorithmus zur Berechnung komplexer FOURIERscher Reihen wird hier zur numerischen Umkehr der Laplace-Transformation verwendet, wobei die ursprüngliche Funktion nach Laguerre-Polynomen entwickelt wird.

The COOLEY-TUKEY algorithm [1] for the machine calculation of complex FOURIER series can be applied advantageously to the numerical inversion of Laplace Transforms. The presently known methods [2—7] of inverting the Laplace Transforms numerically all require N^2 operations, where N is the number of sample points of the transform and an operation is defined as one which consists of one complex multiplication followed by one complex addition. The new method, which is described below, requires $N \log N$ operations. The savings in computer time is clearly substantial. The new method is a modification of that reported by WEEKS [6] and it makes use of the COOLEY-TUKEY algorithm in the evaluation of the coefficients of expansion of the original function.

Let $f(t)$ be the original function, defined over $(0, \infty)$. Let $F(s)$ be its Laplace transform. $f(t)$ and $F(s)$ are related by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{1}$$

and

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds, \quad t \geqslant 0 \tag{2}$$

where c is a suitable constant. The problem is to find $f(t)$ at selected values of t, given $F(s)$ at selected values of s.

[1] This work was done while the author was a Ford Foundation Engineering Resident at IBM Research Center, Yorktown Heights, New York, 1965—1966.

1967 vol 2

Let $f(t)$ be expanded into a series of orthogonal functions:

$$f(t) = e^{ct} \sum_{n=0}^{\infty} a_n \, e^{-\frac{t}{2T}} L_n(t/T), \quad 0 < t < \infty \tag{3}$$

where $L_n(x)$ are Laguerre polynomials of x, and T is a parameter which can be used to control the accuracy of the results. Substitution of (3) into (1) yields

$$F(s) = \sum_{n=0}^{\infty} a_n \frac{\left(s - c - \frac{1}{2T}\right)^n}{\left(s - c + \frac{1}{2T}\right)^{n+1}} \tag{4}$$

Let $F(s)$ be approximated by the first N terms of (4). The coefficients a_n can be found as follows. Rewrite (4):

$$\left(s - c + \frac{1}{2T}\right) F(s) \cong \sum_{n=0}^{N} a_n \left(\frac{s - c - \frac{1}{2T}}{s - c + \frac{1}{2T}} \right)^n \tag{5}$$

Let (5) be evaluated at $s = c + j\,\omega$. We have

$$\left(j\,\omega + \frac{1}{2T}\right) F(c + j\,\omega) \cong \sum_{n=0}^{N} a_n \left(\frac{j\,\omega - \frac{1}{2T}}{j\,\omega + \frac{1}{2T}} \right) \tag{6}$$

Let $e^{j\theta} = \left(j\,\omega - \frac{1}{2T}\right) \Big/ \left(j\,\omega + \frac{1}{2T}\right)$ so that

$$\omega = \left(\frac{1}{2T}\right) \cot(\theta/2) \tag{7}$$

Eq. (6) becomes

$$\left[j\left(\frac{1}{2T}\right) \cot(\theta/2) + \frac{1}{2T} \right] \overline{F}(\theta) \cong \sum_{n=0}^{N} a_n \, e^{jn\theta} \tag{8}$$

where $\overline{F}(\theta)$ is $F(c + j\,\omega)$ with ω replaced by $\left(\frac{1}{2T}\right) \cot(\theta/2)$. The right side of (8) is a complex FOURIER series with a_n real. The COOLEY-TUKEY algorithm can now be used to find a_n. As shown in [1], the number of operations for this purpose is $N \log N$.

The behavior of $\overline{F}(\theta)$ at $\theta = 0$ corresponds to the behavior of $F(c + j\,\omega)$ at $\omega = \infty$. It is easy to show that the left side of (8) at $\theta = 0$ can be evaluated from

$$\frac{\overline{F}(\Delta) + \overline{F}(-\Delta)}{4T} + \frac{\overline{F}(\Delta) - \overline{F}(-\Delta)}{2T\Delta} \tag{9}$$

where Δ is a small quantity. In this way $F(c + j\,\omega)$ need not be evaluated at $\omega = \infty$.

In the evaluation of (8), equally spaced values of θ are chosen. The corresponding values of ω are determined by relation (7). Note that the parameter T controls the spacing of the values of ω at which the left side of (8) is to be evaluated.

A computer program for the new method has been written in Fortran IV. A listing of the program is given in the Appendix.

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Omar Wing
Department of Electrical Engineering
Columbia University, New York
N.Y. 10027, U.S.A.

Appendix

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C      NUMERICAL INVERSION OF LAPLACE TRANSFORMS - - - - - OMAR
C      WING THE TRANSFORM IS DEFINED BY THE USER USING
C      FORTRAN FUNCTION SUBPROGRAM. THE FOLLOWING IS AN
C      EXAMPLE.
C      LET F (S) BE THE TRANSFORM. LET FORG (TIME) BE ITS ORIGINAL
C      FUNCTION.
C      LET F (S) = 1/(S * S + 1) BE THE TRANSFORM WHOSE INVERSE IS
C      DESIRED.
C      THE FUNCTION SUBPROGRAM READS AS FOLLOWS.
C      $ IBFTC FUNCF
C            COMPLEX FUNCTION F (C, W)
C            COMPLEX P
C            P = CMPLX (C, W)
C            F = 1. 0/(P * P + 1. 0)
C            RETURN
C            END
C      NOTE THAT THE FUNCTION NAME IS F.
C      THERE ARE FIVE PARAMETERS TO BE SET.
C      (1) M = AN INTEGER EQUAL TO LOG (N), BASE 2, WHERE N IS THE
C      NUMBER OF SAMPLE POINTS OF THE TRANSFORM. NOT TO
C      EXCEED 512.
C      (2) C = ABSCISSA OF LINE ALONG WHICH INVERSE TRANSFORM IS
C      TO BE EVALUATED. IT IS A REAL CONSTANT GREATER THAN THE
C      REAL PART OF THE RIGHTMOST POLE OF THE TRANSFORM.
```

11*

+ a few pages of program

Introduction

Given

$$F_{\mathrm{I}}(s) = \frac{a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1}{s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{A(s)}{B(s)} \tag{1}$$

where $F_{\mathrm{I}}(s) = \mathscr{L}_{\mathrm{I}}[f(t)]$ is the unilateral Laplace transform of $f(t)$, the initial values of $f(t)$ and its first $(n-1)$ derivatives can be expressed as [1]–[6]

$$f(0) = \Delta_1,$$

and

$$f^{(k-1)}(0) = (-1)^{k-1}\Delta_k, \qquad k = 2, 3, \cdots, n. \tag{2}$$

Specifically, the initial values are evaluated at $t=0+$. $\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_n$ are a set of determinants in terms of a_j and b_j possessing the recursive relations [5]

$$\Delta_1 = a_n,$$

and

$$\Delta_k = \sum_{i=1}^{k-1} (-1)^{i+1} b_i \Delta_{k-i} + (-1)^{k-1} a_{n-k+1}, \qquad k = 2, 3, \cdots, n. \tag{3}$$

In the recent letters [7]–[8], the initial value theorem (IVT) for a bilateral Laplace transform (BLT) is shown to be

$$\lim_{s \to \infty} sF_{\mathrm{II}}(s) = f(0+) - f(0-), \tag{4}$$

where $F_{\mathrm{II}}(s) = \mathscr{L}_{\mathrm{II}}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st} dt$. When $F_{\mathrm{II}}(s)$ is in the form of a rational function as in (1), the IVT for the BLT can be extended as

$$f^{(k-1)}(0+) - f^{(k-1)}(0-) = (-1)^{k-1}\Delta_k, \qquad k = 2, 3, \cdots, n. \tag{5}$$

Proof

Rearrange $sF_{\mathrm{II}}(s)$ as

$$sF_{\mathrm{II}}(s) = \Delta_1 + \frac{-\Delta_2 s^{n-1} + (a_{n-2} - a_n b_2)s^{n-2} + \cdots - \Delta_1 b_n}{B(s)}. \tag{6}$$

By the differentiation rule [1],

$$s^n F_{\mathrm{II}}(s) = \mathscr{L}_{\mathrm{II}}\left[\frac{d^n f(t)}{dt^n}\right]. \tag{7}$$

Recognizing that $f^{(1)}(t)$ has an impulse of strength Δ_1 at $t=0$, apply (4) to the remainder [9] (a proper rational fraction) in (6) to get

$$f^{(1)}(0+) - f^{(1)}(0-) = [\delta^{(1)}(0+) - \delta^{(1)}(0-)]\Delta_1 - \Delta_2. \tag{8}$$

The BLT for $f^{(2)}(t)$ can be expressed as

$$s^2 F_{\mathrm{II}}(s) = \Delta_1 s - \Delta_2 + \frac{\Delta_3 s^{n-1} + [a_{n-3} - a_n b_3 - (a_{n-1} - a_n b_1) b_2] s^{n-2} + \cdots + \Delta_2 b_n}{B(s)}. \tag{9}$$

As before,

$$f^{(2)}(0+) - f^{(2)}(0-) = [\delta^{(2)}(0+) - \delta^{(2)}(0-)]\Delta_1 - [\delta^{(1)}(0+) - \delta^{(1)}(0-)]\Delta_2 + \Delta_3. \tag{10}$$

A generalization of the above results in

$$f^{(j)}(0+) - f^{(j)}(0-) = \sum_{i=0}^{j-1} [\delta^{(j-i)}(0+) - \delta^{(j-i)}(0-)](-1)^i \Delta_{i+1} + (-1)^j \Delta_{j+1}, \qquad j = 2, 3, \cdots, n-1 \tag{11}$$

However, from the theory of distributions [10] it follows that the delta function and all its derivatives are identically zero for $t \neq 0$, including the evaluations at $t=0+$ and $t=0-$. Equation (11), therefore, simplifies to

$$f^{(k-1)}(0+) - f^{(k-1)}(0-) = (-1)^{k-1}\Delta_k. \qquad \text{Q.E.D.}$$

K. R. Rao
Dept. of Elec. Engrg.
University of Texas
Arlington, Tex. 76010

N. Ahmed
Dept. of Elec. Engrg.
Kansas State University
Manhattan, Kan.

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LAGUERRE

Approximate Calculation of Cumulative Probability from a Moment-Generating Function

Abstract—**A numerical method is presented for calculating the cumulative distribution of a positive random variable from its moment-generating function. It involves an expansion of the rectangular function in Laguerre functions. As examples, the cumulative exponential and cumulative Poisson probability functions are approximated.**

A common problem is the calculation of the cumulative probability distribution

$$Q(x) = \int_0^x p(y)\, dy, \qquad 0 < x < \infty, \tag{1}$$

of a positive random variable y of which one knows only the moment-generating function (MGF),

$$\mu(s) = E(e^{ys}) = \int_0^\infty e^{ys} p(y)\, dy, \tag{2}$$

where $p(y)$ is the probability density function (PDF) of y.

In signal detection theory, for instance, y is related to the likelihood ratio, and $1 - Q(x)$ is the false-alarm or detection probability for a decision level x. Often the MGF can be worked out rather easily, but it is impossible to determine $p(y)$ from $\mu(s)$ analytically by, for instance, taking the inverse Laplace transform of $\mu(-s)$ or the inverse Fourier transform of $\mu(i\omega)$.

A technique for calculating $Q(x)$ numerically can be derived by writing (1) as

$$Q(x) = \int_0^\infty R(y/x) p(y)\, dy, \tag{3}$$

where $R(t)$ is the rectangular function

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TABLE I

COEFFICIENTS OF LAGUERRE EXPANSION

m	0	1	2	3	4	5	6	7	8
a_m	2	−2	2	−2	1.9999	−1.9992	1.9956	−1.9800	1.9261
m	9	10	11	12	13	14			
a_m	−1.7766	1.4460	−0.87710	0.15695	0.41416	−0.49039			
m	15	16	17	18	19				
a_m	0.071648	0.33527	−0.22910	−0.18569	0.24351				

TABLE II

EXPONENTIAL DISTRIBUTION

x	0.1	0.3	0.5	1.0	1.5	2.0
Error (%)	0.743	0.561	0.422	−0.351	−0.336	−0.285
x	3.0	4.0	5.0	6.0	8.0	10.0
Error (%)	−0.170	−0.0878	−0.0408	−0.0174	−0.00239	−0.000224

TABLE III

POISSON DISTRIBUTION

x	6	8	10	12	14	16	
Q	0.00763	0.0374	0.1185	0.2676	0.4656	0.6641	
Error (%)	−59.6	−17.6	−2.82	−0.133	0.00493	0.135	
x	18	20	22	24	26	28	30
Q	0.8195	0.9170	0.9673	0.9888	0.99669	0.99914	0.99980
Error (%)	0.241	0.735	1.199	1.379	−0.415	−20.2	72.2

$$R(t) = 1,\ 0 < t < 1; \qquad R(t) = 0,\ t > 1. \tag{4}$$

One expands $R(t)$ in a series of Laguerre functions,[1]

$$R(t) = e^{-kt/2} \sum_{m=0}^{\infty} a_m L_m(kt), \tag{5}$$

where[2]

$$a_m = k\int_0^1 e^{-kt/2} L_m(kt)\,dt = 2e^{-k/2}[L_{m-1}(k) - L_m(k)] - a_{m-1}. \tag{6}$$

The series in (5) is to be truncated at a finite number M of terms.

The cumulative distribution is

$$Q(x) = \sum_{m=0}^{\infty} a_m C_m(x), \tag{7}$$

where the coefficients

$$C_m(x) = \int_0^{\infty} e^{-ky/2x} L_m(ky/x) p(y)\,dy \tag{8}$$

can be expressed in terms of $\mu(-k/2x)$ and its derivatives. In particular,

$$C_0(x) = \mu(-k/2x), \tag{9}$$

and by using the formula[3]

$$L_m(t) = (-1)^m t^m/m! - \sum_{r=1}^{m} (-1)^r \binom{m}{r} L_{m-r}(t). \tag{10}$$

a recurrence relation for $C_m(x)$ is easily obtained,

$$C_m(x) = 2^m(m!)^{-1}\left\{s^m \frac{d^m}{ds^m}[\mu(s)]\right\}_{s=-k/2x} - \sum_{r=1}^{m}(-1)^r\binom{m}{r}C_{m-r}(x), \tag{11}$$

which facilitates numerical computation.

The method was tried out with two very different distributions, the exponential,

$$p(y) = e^{-y},\ y > 0; \qquad p(y) = 0,\ y < 0, \tag{12}$$

whose MGF is

$$\mu(s) = (s-1)^{-1}, \qquad \mathrm{Rl}\, s < 1, \tag{13}$$

and the Poisson,

$$p(y) = e^{-\lambda} \sum_{n=0}^{\infty} \lambda^n \delta(y - n)/n! \tag{14}$$

whose MGF is

$$\mu(s) = \exp[\lambda(e^s - 1)]. \tag{15}$$

First it was necessary to determine the best value of the scale parameter k when M terms are used. This was done by hunting the value of k that yielded the minimum mean-square error

$$\mathscr{E} = 1 - \sum_{m=0}^{M-1} a_m^2 \tag{16}$$

in fitting the truncated version of (5) to $R(t)$. For $M = 20$, we found that $k = 43$ gives a mean-square error $\mathscr{E} = 0.01567$. The coefficients a_m are listed in Table I.

For the exponential PDF we list in Table II the percentage error in $Q(x)$ for $0 < Q(x) < 1/2$ and the percentage error in $1 - Q(x)$ for $1/2 < Q(x) < 1$. The relative error decreases with increasing x.

For the Poisson distribution we evaluated $Q(x)$ by the approximation method for values of x halfway between the integers and compared the results with the Poisson distribution summed from $y = 0$ to the greatest integer in x. Table III lists the percentage errors in $Q(x)$ for $0 < Q(x) < 1/2$ and in $1 - Q(x)$ for $1/2 < Q(x) < 1$. Here $\lambda = 15$.

The accuracy is greatest near the mean and poorest in the tails of the Poisson distribution, and this can be expected in most applications. There exist other approximation methods best suited for the tails of a distribution. For large x, the inverse Laplace transform of $\mu(-s)$ can be approximated by the method of steepest descents.[4] For x near 0, an approximation to $Q(x)$ can be obtained from the asymptotic behavior[5] of $u(-s)$ for large s. The method described here fills the gap.

An alternative method is the Edgeworth series, but it has an asymptotic character that restricts its usefulness.[6] There is an optimum number of terms in the Edgeworth series, and if more are used, the accuracy decreases markedly. Numerical Fourier transformation of $\mu(i\omega)$, followed by numerical integration of the PDF $p(y)$, might be used in some cases, but would hardly be suitable for a discrete random variable like the Poisson-distributed one of our second example.

CARL W. HELSTROM
Dept. of Appl. Electrophys.
University of California
La Jolla, Calif. 92037

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NUMERICAL INVERSION OF THE LAPLACE TRANSFORM USING GENERALISED LAGUERRE POLYNOMIALS

LAGUERRE

R. Piessens, Dr.-Ing., and Maria Branders, M.A.

Indexing terms: *Laplace transforms, Transient response, Polynomials*

Abstract

The calculation of the transient response corresponding to a given frequency response is a problem of numerical inversion of a Laplace transform. Two methods are presented: a very economical method, which is suitable only for a limited class of Laplace transforms, and a general method. FORTRAN programs for both methods are described. The general method is compared with other general methods.

List of principal symbols

p = complex variable
t = independent time variable
$f(t)$ = original function
$F(p)$ = Laplace transform of $f(t)$
$\mathscr{L}^{-1}$ = inverse Laplace operator
$\Gamma(t)$ = gamma function
$L_n^{(a)}(t)$ = generalised Laguerre polynomial of degree n
Si (t) = sine integral
Ci (t) = cosine integral
$J_n(t)$ = Bessel function of the first kind
$I_n(t)$ = modified Bessel function
f.f.t. = fast Fourier transform
D = differential operator

1 Introduction

The main difficulty in applying Laplace-transform techniques is the determination of the original function $f(t)$ from its transform

$$F(p) = \int_0^\infty e^{-pt} f(t)\,dt \qquad (1)$$

In many cases, analytical methods fail and numerical methods must be used. The best known numerical methods for the inversion of the Laplace transform are based on the numerical integration of the Bromwich integral[1-13] or on the expansion of the original function in a series of orthogonal functions, particularly orthogonal exponential functions and Laguerre polynomials. Orthogonal exponential functions are very often used for the calculation of transient responses.[13-21] Even Bellman's method[22, 23] is, in fact, a special case of one of these methods, as has been pointed out by Piessens.[24, 25] The principal reason for the importance of orthogonal exponential functions is that only real values of $F(p)$ are required for calculation of the coefficients of the series expansion of $f(t)$. However, the computation of $f(t)$ from values of $F(p)$ on the real axis is numerically unstable.[18-26] Therefore, if a high degree of accuracy is desired, the calculation must be carried out in multiple precision, or methods must be used which determine the original function from values of the transform in the complex plane or from values of the derivatives of the transform, if these can be easily calculated. For this reason, Laguerre expansions are preferable to expansions in orthogonal exponential functions. This has already been noted by several authors.[13, 18, 21, 27-41]

In this paper, we shall consider an extension using generalised Laguerre polynomials, as proposed by Luke,[13] and shall present new methods for the calculation of the Laguerre coefficients of the original function.

Programs CP77 and 78, first received 9th November 1970 and in revised form 14th June 1971. The program listings and accompanying documentation are held in the IEE Computer-Program Library and are available on application and on payment of charges of £6·60 (CP77) and £6·30 (CP78)
Dr. Piessens and Miss Branders are with the Applied Mathematics Division, University of Leuven, Celestijnenlaan 200B, B 3030 Heverlee, Belgium

2 Description of the method

Assume that $f(t)$ can be expanded in a series

$$f(t) = t^a \sum_{k=0}^{\infty} a_k L_k^{(a)}(t) \qquad (a > -1) \qquad (2)$$

where $L_k^{(a)}(t)$ is the generalised Laguerre polynomial of degree k

$$L_k^{(a)}(t) = \frac{1}{k!} e^t t^{-a} D^k (e^{-t} t^{k+a}) = \sum_{m=0}^{k} (-1)^m \binom{k+a}{k-m} \frac{t^m}{m!} \qquad (3)$$

In eqn. 2, a is a free parameter, the choice of which will be discussed below. For $a = 0$, eqn. 2 is an expansion in ordinary Laguerre polynomials. Coefficients a_k of the series expressed by eqn. 2 are given by

$$a_k = \frac{k!}{\Gamma(k+a+1)} \int_0^\infty e^{-t} f(t) L_k^{(a)}(t)\,dt \qquad (4)$$

or

$$a_k = \frac{k!}{\Gamma(k+a+1)} \sum_{m=0}^{k} (-1)^m \binom{k+a}{k-m} \frac{1}{m!} \int_0^\infty e^{-t} f(t) t^m\,dt \qquad (5)$$

or

$$a_k = \frac{k!}{\Gamma(k+a+1)} \sum_{j=0}^{k} \binom{k+a}{k-j} \frac{1}{j!} \frac{d^j}{dp^j} F(p) \Big|_{p=1} \qquad (6)$$

Therefore, if the Laplace transform of $f(t)$ is known, coefficients a_k of the series expansion of eqn. 2 can be calculated by means of eqn. 6. If eqn. 2 is truncated after N terms, an approximation of the original function is obtained. Eqn. 6 is not suited to numerical calculations. There are other methods for the calculation of a_k. By termwise transformation of eqn. 2, we obtain

$$F(p) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(a+k+1)}{k!} \frac{(p-1)^k}{p^{k+a+1}} \qquad (7)$$

If we consider

$$\phi(z) = \left(\frac{1}{1-z}\right)^{a+1} F\left(\frac{1}{1-z}\right) \qquad (8)$$

eqn. 7 yields

$$\phi(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+a+1)}{k!} z^k \qquad (9)$$

so that

$$a_k = \frac{1}{\Gamma(k+a+1)} \frac{d^k}{dz_k} \phi(z) \Big|_{z=0} \qquad (10)$$

Eqns. 6 and 10 are theoretically equivalent, but eqn. 10 is better suited to numerical calculation, as will be explained in Sections 4 and 5.

The foregoing results can be generalised: if $F(p)$ is the Laplace transform of $f(t)$, and if $f(t)$ can be expanded in a series

$$f(t) = e^{-ct} t^a \sum_{k=0}^{\infty} a_k L_k^{(a)}(bt) \qquad (11)$$

$$\phi^{(a,b,c)}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)}{k!} a_k z^k \quad . \quad . \quad . \quad (12)$$

where

$$\phi^{(a,b,c)}(z) = \left(\frac{b}{1-z}\right)^{a+1} F\left(\frac{b}{1-z} - c\right) \quad . \quad . \quad (13)$$

and thus the a_k's are given by eqn. 10 if $\phi(z)$ is replaced by $\phi^{(a,b,c)}(z)$. Henceforth, we shall omit the superscripts a, b and c, and shall write $\phi(z)$ in all cases.

Parameters a, b and c in eqn. 11 are introduced in an attempt to smooth out any irregularity in $f(t)$ and to accelerate the convergence of eqn. 11. It is very important to choose a so that $t^{-a}f(t)$ can be easily approximated by a polynomial. Therefore parameter a must be so determined that $p^{a+1}F(p)$ is analytic, having no branch point at infinity, so that we can write

$$p^{a+1}F(p) = \sum_{k=0}^{\infty} c_k p^{-k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The optimal value of a is obtained if, in eqn. 14, $c_0 \neq 0$. The advantage of using generalised Laguerre polynomials thus becomes evident. Owing to the introduction of parameter a, a much larger class of Laplace transforms can be efficiently inverted. However, an optimal value of a does not always exist. This will be demonstrated below by examples 5 and 6 in Section 6. The value of c is determinative for the asymptotic behaviour of the truncated series of eqn. 11 for $t \to \infty$. If possible, it is preferable to choose $-c$ equal to the real part of the dominating pole of $F(p)$. The value of b will be discussed in Section 5.

There are two problems: the numerical calculation of coefficients a_k and the evaluation of the truncated Laguerre series. The numerical aspects of the second problem will be considered in Section 3. For the calculation of coefficients a_k, we shall give two methods:

(i) some functions $\phi(z)$ can be easily expanded in a power series through algebraic operations. This will be discussed in Section 4

(ii) for any Laplace transform, the derivatives in eqn. 10 can be evaluated by contour integration in the complex plane. This method is quite general, and will be discussed in Section 5.

3 Evaluation of the truncated Laguerre series

There are two methods for the evaluation of the truncated Laguerre series. If the number of terms of the truncated series is determined at the outset, the summation technique of Smith[42] is very efficient. For generalised Laguerre polynomials, the method is as follows: Let $B_{N+2} = B_{N+1} = 0$, and

$$B_r = a_r + \left(2 - \frac{bt+1-a}{r+1}\right) B_{r+1} - \left(1 - \frac{1-a}{r+a}\right) B_{r+2} \quad . \quad . \quad . \quad . \quad (15)$$

for $r = N, N-1, \ldots, 0$. Then

$$\sum_{k=0}^{N} a_k L_k^{(a)}(bt) = B_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Usually, however, eqn. 11 is calculated with $N+1$ and with $N+L$ terms ($L \simeq N/4$) to control the truncation error. In this case, a direct summation of eqn. 11 is preferable. The Laguerre polynomials are then calculated by the recurrence relationship

$$L_n^{(a)}(t) = (2n + a - 1 - t)L_{n-1}^{(a)}(t) - (n - 1 + a)L_{n-2}^{(a)}(t) \quad . \quad . \quad . \quad . \quad (17)$$

where $n = 1, 2, \ldots$

and $L_{-1}^{(a)}(t) = 0$ and $L_0^{(a)}(t) = 1$

For some types of Laplace transforms, the power-series expansion of the corresponding function $\phi(z)$, given by eqn. 8, is explicitly known, or can be easily obtained by algebraic operations, e.g. by multiplication of known series or, if $F(p)$ is a rational function, by long division.

First, as a simple example, consider the Laplace transform

$$F(p) = p^{-\nu-1}\exp(-up^{-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in which u is an arbitrary positive real number. In conformity with eqn. 14, we choose $a = \nu$. Eqn. 8 then gives

$$\phi(z) = \exp(-u + uz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Therefore, the original function is given by

$$f(t) = t^{\nu}e^{-u} \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(k+\nu+1)} L_k^{(\nu)}(t) \quad . \quad . \quad . \quad (20)$$

From eqn. 20, an interesting result can be obtained. Since it is known that

$$\mathscr{L}^{-1}\{p^{-\nu-1}\exp(-up^{-1})\} = (t/u)^{\nu/2}J_\nu\{2\sqrt{(ut)}\} \quad (21)$$

the following important series expansion for the Besse function of the first kind is obtained:

$$J_\nu(x) = e^{-u}(x/2)^\nu \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(k+\nu+1)} L_k^{(\nu)}(x^2/4u) \quad (22)$$

In the same manner, we can derive

$$I_\nu(x) = e^{u}(x/2)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{u^k}{\Gamma(k+\nu+1)} L_k^{(\nu)}(x^2/4u) \quad . \quad . \quad . \quad . \quad (23)$$

Eqns. 22 and 23 are extensions of series expansions given by Ainsworth and Liu.[43]

Many Laplace transforms can be likewise inverted. We have written a computer program LAGRA (CP77) for the inversion of some types of rational and irrational transforms; namely

$$F(p) = p^\mu \frac{a_0p^m + a_1p^{m-1} + \ldots + a_{m-1}p + a_m}{b_0p^n + b_1p^{n-1} + \ldots + b_{n-1}p + b_n} \quad (24)$$

$$F(p) = p^\mu \frac{\sqrt{(a_0p^m + a_1p^{m-1} + \ldots + a_{m-1}p + a_m)}}{b_0p^n + b_1p^{n-1} + \ldots + b_{n-1}p + b_n} \quad (25)$$

$$F(p) = p^\mu \frac{a_0p^m + a_1p^{m-1} + \ldots + a_{m-1}p + a_m}{\sqrt{(b_0p^{2n} + b_1p^{2n-1} + \ldots + b_{2n-1}p + b_{2n})}} \quad (26)$$

$$F(p) = p^\mu \sqrt{\frac{(a_0p^m + a_1p^{m-1} + \ldots + a_{m-1}p + a_m)}{(b_0p^n + b_1p^{n-1} + \ldots + b_{n-1}p + b_n)}} \quad (27)$$

where μ, a_k and b_k are arbitrary real numbers. It is supposed, however, that $F(p)$ is analytic for $\mathrm{Re}(p) \geqslant 1$.

In the program, the optimal value of a according to eqn. 14 is determined, the coefficients of the expression for $\phi(z)$, given by eqn. 8, are calculated, and finally coefficients a_k are calculated. For the computation of these coefficients, the only operations necessary are long division of two polynomials, root squaring of a series and raising a series to a square. Once coefficients a_k are known, the truncated series of eqn. 2 is evaluated using the recurrence formula given by eqn. 17.

4.1 Examples

All the calculations of these examples were carried out in single precision on an IBM 360/44 computer.

Example 1: Consider a rational Laplace transform. In the case of simple poles, the best method for the inversion of rational transforms is undoubtedly partial-fraction expansion.[44] The determination of multiple poles, on the other hand, is a very difficult task, and in such cases an appropriate method such as that given above is particularly useful.

For the Laplace transform

$$F(p) = \frac{p^4 + 4p^3 + 4p^2 + 4p + 8}{p^5 + 5p^4 + 10p^3 + 10p^2 + 5p + 1} \quad (28)$$

in which a, b and c are free parameters, the coefficients a_k in eqn. 11 are also the coefficients in the power-series expansion

$$\phi^{(a,b,c)}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)}{k!} a_k z^k \quad . \quad . \quad . \quad (12)$$

where

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For the Laplace transform

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the results obtained through LAGRA, using 40 terms in the series, are given in Table 1.

Table 1

NUMERICAL RESULTS OF EXAMPLE 1

t	Exact	LAGRA, 40 terms
2	0·766900	0·766900
4	1·483567	1·483547
6	0·939447	0·939445
10	0·120353	0·120354
14	0·008009	0·008017
16	0·001774	0·001844

Example 2: The transform

$$F(p) = \frac{p\sqrt{(p+1)}}{p^3 + p^2 + p + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

has the original function

$$f(t) = \int_0^t \cos(t-u)\exp(-u)/\sqrt{(\pi u)}du \quad . \quad . \quad (30)$$

which is difficult to calculate. Approximate values obtained through LAGRA, using 40 terms, are given in Table 2

Table 2

NUMERICAL RESULTS OF EXAMPLE 2

t	Exact	LAGRA, 40 terms
2	−0·056675	−0·056678
4	−0·753892	−0·753884
6	0·655745	0·655759
10	−0·826932	−0·826927
14	0·425004	0·424963
20	0·610817	0·610916

Example 3: Laplace transforms which are rational functions in the variable $\sqrt{p}$ are very important in electronics.[45–47] For their inversion, Vlach[12] has proposed a partial-fraction expansion

$$F(p) = \sum_i \frac{A_i}{\sqrt{p} + a_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

followed by termwise inversion. If a_i is complex, however, the complementary error function of a complex argument must be calculated. Although there are many algorithms for this purpose,[48–50] it remains a rather difficult task. Furthermore, there are considerable difficulties in the case of multiple poles. We therefore propose the following approximate procedure. $F(p)$ is split up into a sum of two functions

$$F(p) = \sqrt{(p)}R_1(p) + R_2(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

in which $R_1(p)$ and $R_2(p)$ are rational functions. Both terms of eqn. 32 can be inverted through LAGRA. If desired, $R_2(p)$ can also be inverted by partial-fraction expansion. This procedure is only possible if the poles of $R_1(p)$ are in the halfplane $\mathrm{Re}\,(p) < 1$.

As an example, consider the Laplace transform

$$F(p) = \frac{\sqrt{(p)} + 0{\cdot}5}{p + \sqrt{(p)} + 1{\cdot}25} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The exact original function of eqn. 33 is

$$f(t) = (\pi t)^{-1/2} - u/2 - v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

in which u and v are defined by

$$u + jv = w\{(1 + 0{\cdot}5j)\sqrt{t}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and $$w(z) = e^{-z^2}\left(1 + \frac{2j}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right)$$

Values of u and v have been tabulated by Abramowitz and Stegun[51] and by Faddeeva and Terent'ev.[52]

To enable us to carry out the inversion with the aid of LAGRA, we write

$$F(p) = \frac{\sqrt{(p)}(p + 0{\cdot}75)}{p^2 + 1{\cdot}5p + 1{\cdot}5625} + \frac{-0{\cdot}5p + 0{\cdot}625}{p^2 + 1{\cdot}5p + 1{\cdot}5625} \quad . \quad . \quad . \quad . \quad (36)$$

where each term is to be inverted separately.

The results of this approximation inversion, with 40 terms, are given in Table 3.

Table 3

NUMERICAL RESULTS OF EXAMPLE 3

t	Exact	LAGRA, 40 terms
2	−0·0063098	−0·0063105
4	−0·0102381	−0·0102385
6	−0·0071937	−0·0071940
10	−0·0038493	−0·0038486
14	−0·0024373	−0·0024238
20	−0·0014244	−0·0014556

5 General method for calculation of coefficients

This method is based on the theory of complex functions. If $F(p)$ is analytic for $\mathrm{Re}\,(p) \geqslant b/2 - c$, $\phi(z)$ is analytic in and on the unit circle. We then have

$$\frac{\phi^{(k)}(0)}{k!} = \frac{1}{2\pi j}\int_C \frac{\phi(z)}{z^{k+1}}dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in which C is the circle $|z| = r$ with $r \leqslant 1$.

Substituting

$$z = re^{2\pi jt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

we obtain

$$\frac{\phi^{(k)}(0)}{k!} = r^{-k}\int_0^1 \phi(re^{2\pi jt})e^{-2\pi jkt}dt \quad . \quad . \quad . \quad . \quad (39)$$

$k = 0, 1, 2, \ldots$. Since $z^n\phi(z)$ is analytic in and on C, we have

$$\frac{1}{2\pi j}\int_C z^n\phi(z)dz = \int_0^1 e^{2\pi j(n+1)t}\phi(re^{2\pi jt})dt = 0 \qquad (40)$$

$n = 0, 1, 2, \ldots$. By combining eqns. 39 and 40, we obtain

$$\frac{\phi^{(k)}(0)}{k!} = 2r^{-k}\int_0^1 \mathrm{Re}\,\{\phi(re^{2\pi jt})\}\cos 2\pi ktdt \quad . \quad . \quad (41)$$

and $$\frac{\phi^{(k)}(0)}{k!} = -2jr^{-k}\int_0^1 \mathrm{Im}\,\{\phi(re^{2\pi jt})\}\sin 2\pi ktdt \qquad (42)$$

From eqn. 41, with $r = 1$, can be derived

$$\mathrm{Re}\,\{\phi(e^{j\theta})\} = \sum_{k=0}^{\infty}\frac{\Gamma(a+k+1)}{k!}a_k\cos k\theta \quad . \quad . \quad (43)$$

This result can also be obtained by directly substituting eqn. 38 in eqn. 12.

The problem remaining is therefore the calculation of the Fourier cosine coefficients of the function $\mathrm{Re}\,\{\phi(e^{j\theta})\}$. This can be done by one of the following well known approximate formulas:[53]

$$a_k = \frac{2k!}{\Gamma(a+k+1)M}\sum_{l=0}^{M}\psi(\pi l/M)\cos\frac{\pi lk}{M} \qquad (50)$$

or

$$a_k = \frac{2k!}{\Gamma(a+k+1)(M+1)}\sum_{l=0}^{M}\psi\left(\frac{2l+1}{M+1}\,\frac{\pi}{2}\right)\cos\left(\frac{2l+1}{M+1}\,\frac{k\pi}{2}\right) \quad . \quad . \quad . \quad . \quad (51)$$

where

$$\psi(x) = \mathrm{Re}\left\{\left(j\frac{b}{2}\cot g\frac{x}{2} + \frac{b}{2}\right)^{a+1}F\left(j\frac{b}{2}\cot g\frac{x}{2} - c + \frac{b}{2}\right)\right\} \quad . \quad . \quad . \quad . \quad (52)$$

For the evaluation of eqns. 50 and 51, the fast Fourier transform can be applied, but often the number of terms needed in eqn. 11 is not large, so that use of the f.f.t. is unnecessary. We shall now discuss the choice of free parameter b. It is intuitively evident that the convergence of eqn. 49 is better as the singularities of $\psi(z)$ are further removed from the unit circle or, in other words, as the singularities of $F(p)$ are more removed from the vertical line $\mathrm{Re}\,(p) = b/2 - c$ in the complex plane. Therefore, a large value of b is favourable to a good convergence of the series. But then, only a small number of terms can be used because, using the approximate expressions of eqn. 50 or eqn. 51, the relative error of the coefficients a_k increases greatly with k. If, due to roundoff errors, there is no further reduction of coefficients a_k after $N+1$ terms, the series is truncated and the truncation error ϵ_N is approximately equal to the first neglected term

$$\epsilon_N \simeq a_{N+1} e^{-ct} t^a L_{N+1}^{(a)}(bt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The largest zero ξ of $L_{N+1}^{(a)}(bt)$ is given approximately by (see Tricomi[54])

$$\xi \simeq (4N + 2a + 2)/b$$

Therefore, for $t < \xi$, the error ϵ_N oscillates; but for larger values of t, the error increases greatly. This discussion is only valid in the case of fast convergence of the sequence of coefficients a_k. The conclusion is that, with b increasing, only a small number of terms can be used in eqn. 11, and that the approximation interval is less. Therefore, if the original function must be known in a large interval, we must take a small value of b and calculate more terms in eqn. 11. A computer program LAGUER (CP78), which computes coefficients a_k by means of eqn. 50, has been written. The f.f.t. is not used.

6 Comparison with other methods

The method described in Section 5 requires values of the Laplace transform $F(p)$ for values of p which lie on the imaginary axis or on another vertical line in the complex plane. In the literature, some other methods are known that have the same feature.[8, 10] They are based on the evaluation of

$$f(t) = \frac{2e^{dt}}{\pi} \int_0^\infty \cos \omega t \phi(\omega) d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where $\phi(\omega) = \mathrm{Re}\{F(d + j\omega)\}$

and d is a real number, so that $F(p)$ is analytic for $\mathrm{Re}\,(p) \geqslant d$. The methods differ only in the numerical method used for the calculation of the infinite oscillating integral (eqn. 54). Dubner and Abate[9] have proposed a simple trapezoidal rule. The final inversion formula is

$$f(t) \simeq \frac{e^{dt}}{T} \left\{ \phi(0)/2 + \sum_{k=1}^{M} \phi(k\pi/T) \cos \frac{k\pi t}{T} \right\} \qquad (55)$$

They also provide formulas for the choice of free parameters d and T in eqn. 55.

The integral of eqn. 54 can also be evaluated by a formula given by Clendenin[55] based on the piecewise linear approximation of $\phi(\omega)$. We then obtain

$$f(t) \simeq \frac{2e^{dt}}{\pi} \left(\phi(0) A \sin(th/2) + B \sum_{i=1}^{M-1} \phi(hi) \cos(thi) + \phi(c)[t^{-1} \sin(tc) - A \sin\{t(c - h/2)\}] + R_N \right) \quad (56)$$

where $c = Mh$

h = length of subintervals in which $\phi(\omega)$ is approximated to by a linear function

$A = 2t^{-2}h^{-1} \sin(th/2)$

$B = 4t^{-2}h^{-1} \sin^2(th/2)$

$$R_N = \int_0^\infty \phi(\omega) \cos \omega t d\omega$$

In a practical situation, M must be chosen large enough so that, in expr. 56, R_N is negligible. For some types of Laplace transforms, an asymptotic approximation of $\phi(\omega)$ can be constructed so that R_N can be estimated quite accurately. This has been done by Kowerski[10] for rational Laplace transforms. In the following examples, we shall compare our method (subroutine LAGUER) with that of Dubner and Abate and with that of Clendenin.

6.1 Numerical examples

All calculations of the following examples were carried out in double precision. $M + 1$ indicates the number of evaluations of the Laplace transform, i.e. the number of terms in exprs. 50, 55 and 56. N is the number of terms in the approximation for $f(t)$. Therefore, in our method, N is the number of terms of the truncated Laguerre series, and in both other methods $N = M + 1$. For the other symbols in the Tables, we refer to the text.

6.2 Example 4

In Table 4, the numerical results of the inversion of

$$F(p) = (p^2 + 1)^{-1/2}$$

are given. They illustrate the influence of the value of b.

Table 4

NUMERICAL RESULTS OF EXAMPLE 4

t	Exact	LAGUER, $a = c = 0$, $M = 250$, $N = 190$, $b = 0{\cdot}6$	LAGUER, $a = c = 0$, $M = 20$, $N = 15$, $b = 8$	Dubner and Abate, $M = 800$, $d = 0{\cdot}5$, $T = 20$	Clendenin, $M = 400$, $h = 0{\cdot}1$, $d = 0{\cdot}5$	LAGRA, 110 terms
2	0·2238907791412	0·2238907791411	0·2238907791417	0·223895	0·2229	0·2238907791412
4	−0·3971498098638	−0·3971498098633	−0·3971498108067	−0·397138	−0·3939	−0·3971498098638
8	0·1716508071376	0·1716508071395	0·174119	0·171738	0·1632	0·1716508071376
10	−0·2459357644513	−0·2459357644487	0·461037	−0·245705	−0·2287	−0·2459357644513
20	0·1670246643406	0·1670246643510				0·1670246643406
40	0·0073668905842	0·0073668911301				0·0073668658
60	−0·0914718040891	−0·09147173				−0·09151
80	−0·0697421655122	−0·06971420				
100	0·0199858503042	0·02155037				

For small values of b, very high accuracy can be obtained at the cost of much computation work. To compare LAGUER and LAGRA, results calculated with a double-precision version of LAGRA are given in the last column. Note that, with respect to computation time, LAGRA is much more economical.

6.3 Example 5

For some Laplace transforms, there are no optimal values of a. The obtainable results are less accurate. In Table 5, results are given of the inversion of

$$F(p) = \frac{p \log p}{p^2 + 1}$$

The exact original function is

$$f(t) = -\sin t \,\mathrm{Si}\,(t) - \cos t \,\mathrm{Ci}\,(t)$$

NUMERICAL RESULTS OF EXAMPLE 5

t	Exact	LAGUER, $M = N = 100$, $b = 1$, $a = -0 \cdot 5$, $c = 0$	Dubner and Abate, $M = 200$, $d = 0 \cdot 5$, $T = 20$	Clendenin, $M = 400$, $d = 0 \cdot 5$, $h = 0 \cdot 1$
1	−0·9784	−1·0222	−0·9759	−0·95
2	−1·2838	−1·2833	−1·2775	−1·31
3	−0·1425	−0·1406	−0·131	−0·11
4	1·2385	1·19	1·26	1·21
5	1·5402	1·50	1·57	1·47
6	0·4634	0·57	0·51	0·53
7	−1·0135	−1·11	−0·93	−1·08
8	−1·5396	−1·49	−1·40	−1·40
9	−0·6358	−0·63	−0·41	−0·50
10	0·8640	0·83	1·24	0·45

6.4 Example 6

Sometimes, $F(p)$ can be written as a sum of two or more Laplace transforms which can be inverted separately, each with a different value of a.

For instance,

$$F(p) = p^{-1/2} \exp(-p^{-1/2})$$

can be written as

$$F(p) = p^{-1/2} \cosh(p^{-1/2}) - p^{-1/2} \sinh(p^{-1/2})$$

Here, the first term of the second member can be inverted with $a = 0 \cdot 5$, and the second term with $a = 0$.

The exact original function is

$$f(t) = \frac{1}{2t\sqrt{(\pi t)}} \int_0^\infty u \exp(-u^2/4t) J_0(2\sqrt{u}) du$$

Results are given in Table 6. The methods of Dubner and Abate, and of Clendenin, do not provide reasonable results.

Table 6

NUMERICAL RESULTS OF EXAMPLE 6

t	Exact	LAGUER, $M = 50$
1	−0·010723429	−0·010723401
10	−0·024785984	−0·024785988
20	−0·003081880	−0·003081881
50	0·002719950	0·002719946
100	0·000210929	0·000210934

In these examples, our method offers more accuracy for less computation work.

7 Conclusions

We have given two methods for the numerical inversion of the Laplace transform: the first is applicable only to special types of Laplace transforms, but is very efficient; the second is a general method. The latter is compared with other methods and is found to be very accurate and economical.

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9 Program descriptions

9.1 LAGRA (CP77)

9.1.1 Program details

(*a*) Language: FORTRAN IV
(*b*) Number of variables:
integers: 22
real scalars: 20
arrays: 9
(*c*) Number of statements: 140

9.1.2 Performance guide

(*a*) Computer used: IBM 360/44 of the Computing Centre of the University of Leuven
(*b*) Core-size requirement: 0011EC bytes
(*c*) Output medium: line printer
(*d*) Time: the calculation of 40 terms of the Laguerre expansion and the calculation of the original function of $F(p) = (p^2 + 1)^{-1/2}$ for 15 values of t requires less than 1 s
(*e*) Limitations: the program is applicable only to certain types of Laplace transforms as described in the paper and the poles must lie in the halfplane Re $(p) < 1$
(*f*) Accuracy depends on the Laplace transform t
verted and on the values of t for which the original
must be calculated. In general, accuracy is very h

9.2 LAGUER (CP 78)

9.2.1 Program details

(*a*) Language: FORTRAN IV
(*b*) Number of variables:
integers: 9
real scalars: 20
complex scalars: 4
arrays: 2
(*c*) Special word-length requirements: the progı
written for double-precision arithmetic
(*d*) Number of statements: 68

9.2.2 Performance guide

(*a*) Computer used: IBM 360/44 of the Computing (
of the University of Leuven
(*b*) Core-size required: 0015E8 bytes
(*c*) Output medium: line printer
(*d*) Time: the inversion of $F(p) = (p^2 + 1)^{-1/2}$ for ,
2, . . . , 30, so that the maximal error in this inteı
less than 10^{-5}, requires 5 s
(*e*) Accuracy: depends on the Laplace transform ı
inverted and on the choice of the free parameter
demonstrated in the examples, accuracy can be extre
high

be taken into account in any practical example where some upper bound for the error should be estimated.

HERBERT E. SALZER

Ordnance Corps
Diamond Ordnance Fuze Laboratories
Washington, D. C.

NOTE BY REFEREE

The function $F(p)$ is subject to certain restrictions because it is a Laplace transform. In order for $F(p)$ to be the Laplace transform of the function $f(t)$ given by (1), it is sufficient that $F(p)$ have the form (cf. G. DOETSCH [5]):

$$F(p) = a/p + F_1(p)/p^{1+\delta},$$

where $\delta > 0$, a is a constant, and $F_1(p)$ is analytic and bounded in the half plane $\mathrm{Re}(p) > c$. We assume that this condition is satisfied. Whether this condition is also sufficient for the convergence of the n-point quadrature formula to the true value of $f(t)$ in (1), when n tends to infinity, has not been determined. The author makes use here of the fact that the convergence occurs whenever $F(p)$ is a polynomial in $1/p$ without a constant term; in fact, the quadrature is exact for polynomials of degree not greater than $2n$. G. SZEGÖ [10] has shown that under quite general conditions a Gauss-Jacobi type quadrature formula which converges for polynomials also converges for a much wider class of functions. Unfortunately his theorems do not seem to apply directly to the present case because the integral (1) involves a complex valued weight function which is not of bounded variation.

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6. The shift in notation from $(n+1)$ to n in $A_i^{(n)}$ will cause no confusion after the A_i's have been computed and are ready for use in (6).
7. It was called to the author's attention by H. L. KRALL that $P_n(x) \equiv (-1)^n y_n(x, 1, -1)$ where $y_n(x, a, b)$ are "generalized Bessel polynomials" (see [4]).
8. G. SZEGÖ, *op. cit.*, p. 41–42.
9. Formula (14) holds for $n = 2$ if we define $P_0(x) \equiv 1$.
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On the Improvement of the Solutions to a Set of Simultaneous Linear Equations using the ILLIAC

The basic method used for solving simultaneous linear equations on the University of Illinois' electronic digital computer, the ILLIAC, has already been described in detail by WHEELER and NASH [1]. The routine currently in use on the ILLIAC, programmed by Wheeler [2], makes use of the method of elimination to solve the set of n simultaneous linear equations

$$\sum_{j=0}^{n-1} a_{ij}x_j + a_{in} = 0 \quad i = 0, 1, 2, \cdots, n-1 \tag{1}$$

in a manner very similar to that used by a human solving such a system.

In brief, the procedure used is as follows:

a) The augmented matrix

$$a_{ij} \quad \begin{array}{l} i = 0, 1, 2, \cdots, n-1 \\ j = 0, 1, 2, \cdots, n \end{array} \tag{2}$$

TABLE OF ZEROS AND CHRISTOFFEL NUMBERS—*Continued*

	$p_i^{(n)}$	$1/p_i^{(n)}$	$A_i^{(n)}$	
		$n = 6$		
$i = 1$:	4.03884 8 − 8.34560 0i	.04698 447 +.09708 551i	− 43.516	− 14.015i
$i = 2$:	4.03884 8 + 8.34560 0i	.04698 447 −.09708 551i	− 43.516	+ 14.015i
$i = 3$:	6.47051 3 − 4.90012 1i	.09821 855 +.07438 093i	226.060	+ 305.103i
$i = 4$:	6.47051 3 + 4.90012 1i	.09821 855 −.07438 093i	226.060	− 305.103i
$i = 5$:	7.49064 0 − 1.62149 9i	.12752 426 +.02760 517i	− 185.544	− 917.794i
$i = 6$:	7.49064 0 + 1.62149 9i	.12752 426 −.02760 517i	− 185.544	+ 917.794i
		$n = 7$		
$i = 1$:	4.37869 4 −10.16969 3i	.03571 656 +.08295 315i	− 2.053	+ 68.442i
$i = 2$:	4.37869 4 +10.16969 3i	.03571 656 −.08295 315i	− 2.053	− 68.442i
$i = 3$:	7.14105 5 − 6.62304 6i	.07528 041 +.06981 960i	515.229	− 595.468i
$i = 4$:	7.14105 5 + 6.62304 6i	.07528 041 −.06981 960i	515.229	+ 595.468i
$i = 5$:	8.51183 5 − 3.28101 4i	.10228 557 +.03942 750i	−2490.669	+ 1040.334i
$i = 6$:	8.51183 5 + 3.28101 4i	.10228 557 −.03942 750i	−2490.669	− 1040.334i
$i = 7$:	8.93683 3 + .00000 0i	.11189 646 +.00000 000i	3961.985	+ .000i
		$n = 8$		
$i = 1$:	4.68549 5 −12.01057 8i	.02819 058 +.07226 240i	94.94	− 25.48i
$i = 2$:	4.68549 5 +12.01057 8i	.02819 058 −.07226 240i	94.94	+ 25.48i
$i = 3$:	7.73869 0 − 8.37088 1i	.05954 718 +.06441 172i	−1334.9	− 702. 4i
$i = 4$:	7.73869 0 + 8.37088 1i	.05954 718 −.06441 172i	−1334.9	+ 702. 4i
$i = 5$:	9.40637 0 − 4.96922 0i	.08311 501 +.04390 820i	3848.5	+ 5690. 3i
$i = 6$:	9.40637 0 + 4.96922 0i	.08311 501 −.04390 820i	3848.5	− 5690. 3i
$i = 7$:	10.16944 4 − 1.64920 3i	.09581 390 +.01553 837i	−2613.	−13549. i
$i = 8$:	10.16944 4 + 1.64920 3i	.09581 390 −.01553 837i	−2613.	+13549. i

Table of Zeros and Christoffel Numbers

	$p_i^{(n)}$	$1/p_i^{(n)}$	$A_i^{(n)}$
		$n = 1$	
$i = 1$:	1.00000 000 + .00000 000i	1.00000 0000 +.00000 0000i	1.00000 000 + .00000 000i
		$n = 2$	
$i = 1$:	2.00000 000 − 1.41421 356i	.33333 3333 +.23570 2260i	− 1.00000 00 − 3.53553 39i
$i = 2$:	2.00000 000 + 1.41421 356i	.33333 3333 −.23570 2260i	− 1.00000 00 + 3.53553 39i
		$n = 3$	
$i = 1$:	2.68108 288 − 3.05043 020i	.16255 5585 +.18494 9324i	− 7.64874 9 + 4.17164 0i
$i = 2$:	2.68108 288 + 3.05043 020i	.16255 5585 −.18494 9324i	− 7.64874 9 − 4.17164 0i
$i = 3$:	3.63783 425 + .00000 000i	.27488 8830 +.00000 0000i	18.29749 8 + .00000 0i
		$n = 4$	
$i = 1$:	3.21280 69 − 4.77308 75i	.09705 0482 +.14418 2470i	11.3015 + 12.4717i
$i = 2$:	3.21280 69 + 4.77308 75i	.09705 0482 −.14418 2470i	11.3015 − 12.4717i
$i = 3$:	4.78719 30 − 1.56747 65i	.18866 3804 +.06177 4421i	− 13.30154 − 60.07173i
$i = 4$:	4.78719 30 + 1.56747 65i	.18866 3804 −.06177 4421i	− 13.30154 + 60.07173i
		$n = 5$	
$i = 1$:	3.65569 43 − 6.54373 69i	.06506 5779 +.11646 8528i	15.8268 − 24.1256i
$i = 2$:	3.65569 43 + 6.54373 69i	.06506 5779 −.11646 8528i	15.8268 + 24.1256i
$i = 3$:	5.70095 33 − 3.21026 56i	.13317 9077 +.07499 4511i	− 149.9984 + 68.0423i
$i = 4$:	5.70095 33 + 3.21026 56i	.13317 9077 −.07499 4511i	− 149.9984 − 68.0423i
$i = 5$:	6.28670 47 + .00000 00i	.15906 5845 +.00000 0000i	273.3433 + .0000i

where

$$a_0 = (-1)^n, \quad \text{and} \quad r a_r = -\left(n^2 - \overline{r-1}^2\right) a_{r-1}, \quad \text{for} \quad r > 0.$$

Working backwards from (12″), by equating coefficients of x^{r-1}, one sees that (12″) must arise from (16).

VIII. Explicit expressions for polynomials. Because these polynomials $P_n(x)$ are of fundamental importance, and their role in the inverse Laplace transform is comparable to the role of the Laguerre polynomials in the direct Laplace transform, their explicit expressions are given below for $n = 1(1)12$:

$$P_1(x) = x - 1$$

$$P_2(x) = 6x^2 - 4x + 1$$

$$P_3(x) = 60x^3 - 36x^2 + 9x - 1$$

$$P_4(x) = 840x^4 - 480x^3 + 120x^2 - 16x + 1$$

$$P_5(x) = 15120x^5 - 8400x^4 + 2100x^3 - 300x^2 + 25x - 1$$

$$P_6(x) = 3\ 32640x^6 - 1\ 81440x^5 + 45360x^4 - 6720x^3 + 630x^2 - 36x + 1$$

$$P_7(x) = 86\ 48640x^7 - 46\ 56960x^6 + 11\ 64240x^5 - 1\ 76400x^4 + 17640x^3 - 1176x^2 + 49x - 1$$

$$P_8(x) = 2594\ 59200x^8 - 1383\ 78240x^7 + 345\ 94560x^6 - 53\ 22240x^5 + 5\ 54400x^4 - 40320x^3 + 2016x^2 - 64x + 1$$

$$P_9(x) = 88216\ 12800x^9 - 46702\ 65600x^8 + 11675\ 66400x^7 - 1816\ 21440x^6 + 194\ 59440x^5 - 14\ 96880x^4 + 83160x^3 - 3240x^2 + 81x - 1$$

$$P_{10}(x) = 33\ 52212\ 86400x^{10} - 17\ 64322\ 56000x^9 + 4\ 41080\ 64000x^8 - 69189\ 12000x^7 + 7567\ 56000x^6 - 605\ 40480x^5 + 36\ 03600x^4 - 1\ 58400x^3 + 4950x^2 - 100x + 1$$

$$P_{11}(x) = 1407\ 92940\ 28800x^{11} - 737\ 48683\ 00800x^{10} + 184\ 37170\ 75200x^9 - 29\ 11132\ 22400x^8 + 3\ 23459\ 13600x^7 - 26637\ 81120x^6 + 1664\ 86320x^5 - 79\ 27920x^4 + 2\ 83140x^3 - 7260x^2 + 121x - 1$$

$$P_{12}(x) = 64764\ 75253\ 24800x^{12} - 33790\ 30566\ 91200x^{11} + 8447\ 57641\ 72800x^{10} - 1340\ 88514\ 56000x^9 + 150\ 84957\ 88800x^8 - 12\ 70312\ 24320x^7 + 82335\ 05280x^6 - 4151\ 34720x^5 + 162\ 16200x^4 - 4\ 80480x^3 + 10296x^2 - 144x + 1.$$

IX. Zeros and Christoffel numbers. In the numerical table below there are given the values of the reciprocals of the zeros of $P_n(x)$ or $p_i^{(n)}$, the zeros of $P_n(x)$, or $1/p_i^{(n)}$, and the corresponding Christoffel numbers $A_i^{(n)}$, for $n = 1(1)8$. Use of these quantities in the quadrature formula (6) above can give theoretically exact accuracy for any polynomial in $1/p$ (with no constant term) up to the 16th degree. However, the fact that these tabulated values of $p_i^{(n)}$, $1/p_i^{(n)}$ and $A_i^{(n)}$ are correct to only about a unit in the last significant figure that is given, must integral coefficients, or, in other words, that

$$2P_n(x) + (2n+1)P_{n-1}(x) \equiv 0 \pmod{2n-1}.$$

This last congruence, by (14), is equivalent to

$$\frac{4}{2n-3}P_{n-1}(x) + \frac{2(2n-1)}{2n-3}P_{n-2}(x) + (2n+1)P_{n-1}(x) \equiv 0 \pmod{2n-1},$$

or to

$$\frac{(2n-1)^2}{2n-3}P_{n-1}(x) + \frac{2(2n-1)}{2n-3}P_{n-2}(x) \equiv 0 \pmod{2n-1},$$

which in turn is expressible as

$$(2n-1)\left[\frac{(2n-1)P_{n-1}(x) + 2P_{n-2}(x)}{2n-3}\right] \equiv 0 \pmod{2n-1},$$

or

$$(2n-1)\left[\frac{(2 + (2n-3)P_{n-1}(x) + (2n-1-(2n-3))P_{n-2}(x)}{2n-3}\right] \equiv 0 \pmod{2n-1}.$$

But under the assumptions that (15) holds for $m = n$, and that $P_m(x), m \leq n-1$, has integral coefficients, the last quantity in brackets is a polynomial with integral coefficients, which shows that the last congruence is satisfied identically in x. Thus (15) holds for $m = n+1$ and $P_{n+1}(x)$ has integral coefficients. We proceed in this way to every n. There is a slight subtlety in the argument of this induction in the sense that the integral coefficients of $P_m(x)$ up to $m = n-1$ only are needed to go from $m = n$ to $m = n+1$ in (15), but then use is made of the integral coefficients of $P_n(x)$ in using (14) with $n+1$ in place of n.

VII. **Differential equation.** It is easy to show that $P_n(x)$ satisfies the differential equation

(16) $$x^2P_n''(x) + (x-1)P_n'(x) - n^2P_n(x) = 0.$$

Thus one merely expresses (12) in the form

(12′) $$P_n(x) = (-1)^n\left[1 + \sum_{r=1}^{n}\frac{(-1)^r n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-\overline{r-1}^2)\,x^r}{r!}\right],$$

and then observes that (12′) is equivalent to the automatically terminating "infinite series."

(12″) $$P_n(x) = \sum_{r=0}^{\infty} a_r x^r,$$

explicit formula for $P_n(1/p)$ in (12). For, in view of (8), it suffices to consider only

$$\text{(H)} \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^p}{p}\frac{(-1)^{2n}}{p^n}\binom{2n-1}{n} n!P_n\left(\frac{1}{p}\right) dp,$$

or

$$\text{(I)} \qquad (-1)^{2n}\binom{2n-1}{n} n! \sum_{r=0}^{n} \frac{(-1)^{2n-r}\binom{n}{r}\binom{2n-r-1}{n-r}(n-r)!}{(2n-r)!},$$

or

$$\text{(J)} \qquad \binom{2n-1}{n} n! \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\cdots(n-r+1)}{r!} \frac{(2n-r-1)\cdots(n+1)n}{(2n-r)(2n-r-1)\cdots(n+1)n!},$$

which, after cancellations, is written as

$$\text{(K)} \qquad \sum_{r=0}^{n} (-1)^r \frac{(2n-1)(2n-2)\cdots n}{(n-r)!n!} \frac{n}{2n-r} \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!},$$

or

$$\text{(L)} \qquad n\sum_{r=0}^{n} (-1)^r \frac{1}{2n-r} \frac{(2n-1)(2n-2)\cdots n}{(n-r)!r!},$$

and this, in turn, is expressible in the form

$$\text{(M)} \qquad \tfrac{1}{2}(-1)^n \times \sum_{r=0}^{n} \frac{(2n-0)(2n-1)(2n-2)\cdots(2n-[r-1])(2n-[r+1])\cdots(2n-n)}{(r-0)(r-1)\cdots(r-[r-1])(r-[r+1])\cdots(r-n)}$$

In (M), the $\frac{1}{2}(-1)^n$ is multiplied by the sum of the coefficients of the Lagrangian interpolation polynomial for the $(n+1)$ points $0, 1, \cdots, n$, for the variable equal to $2n$. But that sum is identically equal to 1, i.e., for any value, $2n$ or otherwise. Thus we obtain once more $\frac{1}{2}(-1)^n$ for the normalization.

VI. Integral coefficients. It may be of interest to show that (14) alone, without any knowledge of (10), implies that $P_n(x)$ has integral coefficients. We prove this by noting that $P_{n+1}(x)$ will have integral coefficients if $P_m(x)$, $m \leq n$, has integral coefficients and the following identical polynomial congruence holds for $m = n + 1$:

$$\text{(15)} \qquad 2P_{m-1}(x) + (2m-1)P_{m-2}(x) \equiv 0 \pmod{2m-3}.$$

Now the existence of integral coefficients of $P_m(x)$ and congruence (15) can be verified for the first few values of m. We then show that if (15) holds for some particular $m = n$, it holds for $m = n + 1$, provided $P_m(x)$, $m \leq n - 1$, has

7. Recurrence formula. It is easy to obtain the recurrence relation for the polynomials $P_n(x)$ by employing a fundamental theorem about the existence of a recurrence formula connecting any three successive orthogonal polynomials (G. Szegö [8]), namely,

$$P_n(x) = (a_n x + b_n)P_{n-1}(x) + c_n P_{n-2}(x). \tag{13}$$

Then a_n is immediately seen to be $4n-2$. Equating constant terms in (13), one finds $c_n = b_n + 1$, and after substitution into the equation derived from the coefficients of x, one obtains

$$b_n = \frac{2}{2n-3}, \quad c_n = \frac{2n-1}{2n-3},$$

so that the recurrence formula satisfied by $P_n(x)$ is seen to be [9]

$$(2n-3)P_n(x) = [(4n-2)(2n-3)x + 2]P_{n-1}(x) + (2n-1)P_{n-2}(x), \quad \text{for} \quad n \geq 3. \tag{14}$$

From (14) and (8) only, without making use of (10), one can again find the normalization factor given in (11), through the following inductive argument: Multiply (14) by $P_{n-2}(x)$ and then operate with

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} \cdots dp$$

to obtain (making use of (8)):

$$0 = \frac{(4n-2)(2n-3)}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} P_{n-1}\left(\frac{1}{p}\right)\cdot\frac{1}{p}P_{n-2}\left(\frac{1}{p}\right) dp$$
$$+ 0 + \frac{(2n-1)}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[P_{n-2}\left(\frac{1}{p}\right)\right]^2 dp.$$

Denoting the left member of (11) by F_n, still making use of (8) to replace in the first of the above integrals $(1/p)P_{n-2}(1/p)$ by

$$\frac{1}{a_{n-1}} P_{n-1}\left(\frac{1}{p}\right) = \frac{1}{4n-6} P_{n-1}\left(\frac{1}{p}\right),$$

one now obtains

$$0 = \frac{(4n-2)(2n-3)}{4n-6} F_{n-1} + (2n-1)F_{n-2},$$

or $F_{n-1} = -F_{n-2}$. Since $F_1 = -\frac{1}{2}$, (11) follows by induction.

The normalization given in (11) can be seen in a third way, directly from the of s between 0 and r, would always be eventually annulled because the initially occurring differential operator d^{m-r+n}/dp^{m-r+n} is of order $m - r + n > n - 1 - s$ even for the highest value of $n - 1 - s$ when $s = 0$ (due to $m - r + n > n - 1$ for every r between 0 and m). Thus (E) vanishes, which proves (10), and establishes at the same time that this normalization yields all integral coefficients for $P_n(1/p)$.

IV. Normalization factor. To obtain the normalization factor, which turns out to be given by

$$\text{(11)} \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[P_n\left(\frac{1}{p}\right)\right]^2 dp = \tfrac{1}{2}(-1)^n,$$

we repeat the preceding argument for $m = n$ and now notice that in the final integral (E) the lowest power of p^{n-1-s} will survive the integration by parts, because it is equal to $1/p$. Retaining in the double summation in (E) only the single non-vanishing term $s = r = m = n$, we get

$$\text{(F)} \qquad \frac{(-1)^{2n+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty}\binom{2n-1}{n} n!\frac{1}{p}\frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)dp,$$

which is integrated by parts n times, the integrated part always vanishing, to give

$$\text{(G)} \qquad \frac{(-1)^{2n+2n}}{2\pi j}\binom{2n-1}{n} n!\int_{c-j\infty}^{c+j\infty} e^p\frac{(-1)^n\cdot 2\cdot 3\cdots(n-1)n}{p^{2n+1}}dp.$$

But (G) is

$$(-1)^n\binom{2n-1}{n} n!n!\frac{1}{(2n)!} = \frac{(-1)^n(2n-1)(2n-2)\cdots n}{n!}n!n!\frac{1}{(2n)!}$$

which reduces to $(-1)^n n/2n$ or $\frac{1}{2}(-1)^n$, thus proving (11).

From (10), the explicit formula for $P_n(1/p)$ is seen to be [7]

$$\text{(12)} \quad P_n\left(\frac{1}{p}\right) = (-1)^n\left[\frac{(-1)^n\binom{2n-1}{n}n!}{p^n} + \frac{(-1)^{n-1}\binom{n}{1}\binom{2n-2}{n-1}(n-1)!}{p^{n-1}} + \frac{(-1)^{n-2}\binom{n}{2}\binom{2n-3}{n-2}(n-2)!}{p^{n-2}} + \cdots + \frac{(-1)^{n-r}\binom{n}{r}\binom{2n-r-1}{n-r}(n-r)!}{p^{n-r}} + \cdots + \frac{n^2(-1)^1}{p} + (-1)^0\right]$$

$P_n(1/p)$ has the following more elegant definition:

$$P_n\left(\frac{1}{p}\right) = (-1)^n e^{-p} p^n \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right). \tag{10}$$

That (10) yields the leading coefficient of $1/p^n$ in $P_n(1/p)$, namely

$$\begin{cases} 1, \quad \text{for} \quad n = 1, \\ (4n-2)(4n-6)\cdots 6, \quad \text{for} \quad n \geq 2, \end{cases}$$

is obvious by induction. To prove the orthogonality property, or (8), it suffices to prove the vanishing of

$$\text{(A)} \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[(-1)^n e^{-p} p^n \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\right]\left[(-1)^m e^{-p} p^m \frac{d^m}{dp^m}\left(\frac{e^p}{p^m}\right)\right] dp$$

for $m < n$. This last expression is written as

$$\text{(B)} \qquad \frac{(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{-p} p^{m+n-1} \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\frac{d^m}{dp^m}\left(\frac{e^p}{p^m}\right) dp,$$

and after integrating by parts m times, noting that the integrated parts always vanish, we have

$$\text{(C)} \qquad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{d^m}{dp^m}\left[e^{-p} p^{m+n-1} \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\right]\frac{e^p}{p^m}\, dp,$$

which by LEIBNITZ's rule is expressible as

$$\text{(D)} \qquad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \sum_{r=0}^{m} \binom{m}{r} \frac{d^r}{dp^r}\left(e^{-p} p^{m+n-1}\right) \frac{d^{m-r+n}}{dp^{m-r+n}}\left(\frac{e^p}{p^n}\right)\frac{e^p}{p^m}\, dp.$$

Application of Leibnitz's rule a second time to

$$\frac{d^r}{dp^r}\left(e^{-p} p^{m+n-1}\right)$$

in the above and cancellation of e^p/p^m, yields

$$\text{(E)} \qquad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \sum_{r=0}^{m} \binom{m}{r}\left[\sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}\binom{m+n-1}{s} s!\, p^{n-1-s}\right] \times \frac{d^{m-r+n}}{dp^{m-r+n}}\left(\frac{e^p}{p^n}\right) dp.$$

Now we integrate by parts $(m - r + n)$ times each term of the above double summation. The integrated part will always vanish since it will have a factor of $1/p$ to at least the first power. Furthermore, at some stage in the partial integration of each term, that stage varying with the term, the integral part will also vanish if $m < n$. This last follows because the lowest power of p^{n-1-s} is positive or zero, since s can equal at the most r which can equal at the most $m \leq n - 1$. Then in the integration by parts the positive or zero power p^{n-1-s}, for each value

Hence the points $1/p_i$, now denoted by $1/p_i^{(n)}$, are the zeros of a certain set of orthogonal polynomials in the variable $1/p$.

The condition of orthogonality (8) is also mathematically equivalent, in terms of actual polynomials (by setting $x = 1/p$), to having a polynomial of the n^{th} degree $q_n(x)$ which is orthogonal to any $\rho_{n-1}(x)$, with weight function $e^{1/x}/x$, where the path of integration is a circle of radius $1/2c$ whose center is at $(1/2c, 0)$.

If the polynomial $p_n(1/p)$ is written as

$$\left(\frac{1}{p}\right)^n + b_{n-1}\left(\frac{1}{p}\right)^{n-1} + b_{n-2}\left(\frac{1}{p}\right)^{n-2} + \cdots + b_1\left(\frac{1}{p}\right) + b_0,$$

the determination of b_i, $i = 0, 1, \cdots, n-1$, to satisfy the conditions of orthogonality (8), making use of

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^p}{p^{m+1}}\,dp = \frac{1}{m!}$$

is in the solution of this system of linear equations:

$$(9')\quad \begin{cases} \dfrac{1}{n!} + \dfrac{b_{n-1}}{(n-1)!} + \dfrac{b_{n-2}}{(n-2)!} + \cdots + \dfrac{b_1}{1!} + \dfrac{b_0}{0!} = 0 \\[2ex] \dfrac{1}{(n+1)!} + \dfrac{b_{n-1}}{n!} + \dfrac{b_{n-2}}{(n-1)!} + \cdots + \dfrac{b_1}{2!} + \dfrac{b_0}{1!} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{(2n-1)!} + \dfrac{b_{n-1}}{(2n-2)!} + \dfrac{b_{n-2}}{(2n-3)!} + \cdots + \dfrac{b_1}{n!} + \dfrac{b_0}{(n-1)!} = 0. \end{cases}$$

For numerical work it is somewhat easier to solve (9′) in the form

$$(9)\quad \begin{cases} 1 + nb_{n-1} + n(n-1)b_{n-2} + \cdots + n!b_1 + n!b_0 = 0 \\ 1 + (n+1)b_{n-1} + (n+1)nb_{n-2} + \cdots \\ \qquad\qquad + (n+1)\cdots 3b_1 + (n+1)\cdots 2b_0 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 + (2n-1)b_{n-1} + (2n-1)(2n-2)b_{n-2} + \cdots \\ \qquad\qquad + (2n-1)\cdots(n+1)b_1 + (2n-1)\cdots nb_0 = 0. \end{cases}$$

III. Explicit expression for orthogonal polynomials. It is convenient to normalize the polynomials $p_n(x)$, where $x \equiv 1/p$, by multiplying $p_n(x)$, for $n \geq 2$, by $(4n-2)(4n-6)\cdots 6$. This normalization produces polynomials with all coefficients integral (proven below) and it is not the usual normalization by multiplication by

$$\left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left\{p_n\left(\frac{1}{p}\right)\right\}^2 dp\right]^{-\frac{1}{2}}.$$

Denoting $(4n-2)(4n-6)\cdots 6p_n(1/p)$ by $P_n(1/p)$ for $n \geq 2$, and $p_1(1/p)$ by $P_1(1/p)$, one can avoid the labor of solving (9′) or (9) directly by showing that where on the right hand side of (2)

$$L_i^{(n+1)}\left(\frac{1}{p}\right) \equiv \prod_{k=1}^{n+1}{}' \left(\frac{1}{p} - \frac{1}{p_k}\right) \bigg/ \prod_{k=1}^{n+1}{}' \left(\frac{1}{p_i} - \frac{1}{p_k}\right), \tag{3}$$

the $\prod'$ denoting the absence of $k = i$. In (2), p_{n+1} is ∞, so that there is actually no $(n+1)$-th term in (2) and $L_{n+1}^{(n+1)}(1/p)$ is not used. (The summation in (2) is written with $n+1$ instead of n to avoid confusion with the $(n-1)$-th degree coefficients $L_i^{(n)}(1/p)$ which differ from the $L_i^{(n+1)}(1/p)$ by not having the factor p_i/p.)

Following the method in G. SZEGÖ [3], we consider the $(2n)$-th degree polynomial in $1/p$, namely, $\rho_{2n}(1/p) - L^{(n+1)}(1/p)$ which vanishes at $1/p = 0$, $1/p_i$, $i = 1, 2, \cdots, n$, and thus has

$$\frac{1}{p} p_n\left(\frac{1}{p}\right) \equiv \frac{1}{p} \prod_{i=1}^{n} \left(\frac{1}{p} - \frac{1}{p_i}\right)$$

as a factor. Writing

$$\rho_{2n}\left(\frac{1}{p}\right) = L^{(n+1)}\left(\frac{1}{p}\right) + \frac{1}{p} p_n\left(\frac{1}{p}\right) r_{n-1}\left(\frac{1}{p}\right), \tag{4}$$

it follows that

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p \rho_{2n}\left(\frac{1}{p}\right) dp = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p L^{(n+1)}\left(\frac{1}{p}\right) dp + \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} p_n\left(\frac{1}{p}\right) r_{n-1}\left(\frac{1}{p}\right) dp. \tag{5}$$

Thus if the second term in the right member of (5) always vanishes, (5) will be an n-point quadrature formula that is exact for any $(2n)$-th degree polynomial in $1/p$ without a constant term, namely,

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p \rho_{2n}\left(\frac{1}{p}\right) dp = \sum_{i=1}^{n} A_i^{(n)} \rho_{2n}\left(\frac{1}{p_i}\right), \tag{6}$$

where the "Christoffel numbers" $A_i^{(n)}$ are given by [6]

$$A_i^{(n)} \equiv \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p L_i^{(n+1)}\left(\frac{1}{p}\right) dp. \tag{7}$$

A sufficient condition for (6) to hold is obviously the "orthogonality" of $(1/p)p_n(1/p)$ with respect to any arbitrary $\rho_{n-1}(1/p)$, namely,

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} p_n\left(\frac{1}{p}\right) \left(\frac{1}{p}\right)^i dp = 0, \quad i = 0, 1, \cdots, n-1. \tag{8}$$

The necessity of (8) is also obvious from (6) by choosing

$$\rho_{2n}(1/p) = (1/p) p_n(1/p) \rho_{n-1}(1/p)$$

where $\rho_{n-1}(1/p)$ is any arbitrary polynomial in $1/p$ of the $(n-1)$-th degree.

only to solution in series or numerical integration, with an $F(p)$ that is given as a tabulated function of p. Also, the solution $F(p)$ might be given explicitly in closed form as a combination of integrals of such complicated analytic expressions that it might be easier to evaluate it for different numerical values of p than to find its poles, residues, and branch points.

The purpose of this present article is to discuss the properties of a new set of orthogonal polynomials which can be the basis for convenient formulas for approximating $f(t)$ in (1) for different positive values of t when one has an $F(p)$ that is too complicated to show its analytic character, but which can be calculated for any p.

All further discussion will now be for $F(p)$ assumed to be exactly of the form $\sum_{r=1}^{m} \frac{a_r}{p^r}$, i.e., a polynomial in $1/p$ without a constant term.

To obtain a definite integral without a parameter t in the exponential term, which is the "weight function," let $pt = u$ in (1), so that we obtain

$$(1') \qquad f(t) = \frac{1}{2\pi j t} \int_{c_1 - j\infty}^{c_1 + j\infty} e^u F\left(\frac{u}{t}\right) du,$$

where $F\left(\frac{u}{t}\right)$ is still a polynomial in $1/u$, without a constant term.

II. Use of orthogonal polynomials. At this point one may recall the application of the theory of orthogonal polynomials to quadrature formulas for definite integrals where the integrand is the product of a preassigned weight function and a polynomial $P(t)$. There it is possible to employ the value of $P(t)$ at n fixed irregularly spaced points t_i, $i = 1, 2, \cdots, n$, such that the resulting quadrature formula is exact when $P(t)$ is any arbitrary polynomial of $(2n - 1)$-th degree. Thus for the direct Laplace transform of $P(t)$, namely $\int_0^\infty e^{-pt} P(t) dt$, which is essentially $\int_0^\infty e^{-t} Q(t) dt$ for polynomial $Q(t)$, the points t_i are taken equal to the zeros of the Laguerre polynomials, which have been tabulated extensively (H. E. SALZER and R. ZUCKER [2]). In the present case, even though we are not dealing with a polynomial in p, we can still solve the problem of finding a Gaussian-type quadrature formula for (1′) of approximately double the degree of accuracy of an ordinary quadrature formula based upon the same number of equally spaced points.

Thus let $\rho_{2n}(1/p)$ be any arbitrary $(2n)$-th degree polynomial in the variable $1/p$, which vanishes at $1/p = 0$. Consider n distinct points $1/p_i$, $i = 1, 2, \cdots, n$ other than $1/p = 0$ and construct the $(n + 1)$-point Lagrangian polynomial approximation (of the n^{th} degree in $1/p$), to $\rho_{2n}(1/p)$, based upon the points $1/p_i$, $i = 1, 2, \cdots, n$ and $1/p = 0$. The $(n + 1)$th point $1/p = 0$ is needed in order to provide for the property that $\rho_{2n}(1/p)$ vanishes at $p = \infty$. We have for this polynomial approximation $L^{(n+1)}(1/p)$ the explicit expression

$$(2) \qquad L^{(n+1)}\left(\frac{1}{p}\right) = \sum_{i=1}^{n+1} L_i^{(n+1)}\left(\frac{1}{p}\right) \rho_{2n}\left(\frac{1}{p_i}\right),$$

formula that is exact when ρ_{2n} is any arbitrary polynomial of the $(2n)$th degree in $x \equiv 1/p$ without a constant term, namely: (2) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p \rho_{2n}(1/p)dp = \sum_{i=1}^{n} A_i^{(n)}\rho_{2n}(1/p_i)$. In (2), $x_i \equiv 1/p_i$ are the zeros of the orthogonal polynomials $p_n(x) \equiv \prod_{i=1}^{n}(x - x_i)$ where (3) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p(1/p)p_n(1/p)(1/p)^i dp = 0, i = 0, 1, \cdots, n-1$ and $A_i^{(n)}$ correspond to the CHRISTOFFEL numbers. The normalization $P_n(1/p) \equiv (4n-2)(4n-6)\cdots 6p_n(1/p)$, $n \geq 2$, produces all integral coefficients. $P_n(1/p)$ is proven to be $(-1)^n e^{-p}p^n d^n(e^p/p^n)/dp^n$. The normalization factor is proved, in three different ways, to be given by (4) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p(1/p)[P_n(1/p)]^2 dp = \frac{1}{2}(-1)^n$. Proofs are given for the recurrence formula (5) $(2n-3)P_n(x) = [(4n-2)(2n-3)x+2]P_{n-1}(x) + (2n-1)P_{n-2}(x)$, for $n \geq 3$, and the differential equation (6) $x^2P_n''(x) + (x-1)P_n'(x) - n^2P_n(x) = 0$. The quantities $p_i^{(n)}$, $1/p_i^{(n)}$ and $A_i^{(n)}$ were computed, mostly to 6S − 8S, for $i = 1(1)n$, $n = 1(1)8$.

I. Introduction: Occurrence of inverse Laplace transforms. For a given function of p, $F(p)$, which is the direct Laplace transform of some unknown function $f(t)$, for $t > 0$, one usually finds the $f(t)$ from the following explicit expression:

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{pt}F(p)dp. \tag{1}$$

Formula (1) is known as the inverse Laplace transform of $F(p)$. In (1) the quantity c is a real constant ± 0 that is greater than the real part of all the singular points of $F(p)$. In practice c is usually positive, but c can be negative as long as for $f(t)$ satisfying Dirichlet's conditions in any finite positive interval the integral $\int_0^\infty e^{-ct}f(t)dt$ is absolutely convergent (H. S. CARSLAW and J. C. JAEGER [1]).

A note by the referee follows this paper and indicates relations between the present work and work published elsewhere.

The examples treated in most textbooks on operational calculus and Laplace transforms contain such functions $F(p)$ that their poles and branch points (and residues also) are obtainable without too much difficulty, and the inversion integral in (1) is evaluated by suitable deformation of the path of integration, and the use of Cauchy's theorem. But there are countless other examples where $F(p)$ might be too complicated to yield explicit information about the location and nature of its singularities without a prohibitive amount of labor. For instance, one will recall that in most textbook examples treating the solution of ordinary and partial differential equations by operational means, the original system of differential equations is transformed into a system whose solution $F(p)$ is usually some known elementary function or a very extensively tabulated function of a simple differential equation (like a Bessel function), so that its analytic character and singularities are well known. But in actual practice one might not be fortunate enough to obtain such a comparatively simple $F(p)$. Thus the transformed differential equations might not yield a known function. Instead it might be amenable four digits of the final result. The result has been tabulated to 3089D; the final digit is unrounded.

Running time for the 3093D was approximately thirteen minutes. The programming takes account of the number of zeros generated to the right of the decimal point in each factor, so that the number of operations required for each term in the series decreases. This leads to the following statement—if the time to compute π to m digits is t units, then the time to produce km digits is roughly k^2t units; this holds true as long as the calculation is contained in high-speed storage.

The following table gives a count of each of the digits in π.

(1)	(2) 1–3090	(3) 1–2036	(4) 2037–3090	(5) (4)/(3)
0	269	184	85	.46
1	315	213	102	.47
2	314	210	104	.50
3	276	191	85	.45
4	322	198	124	.63
5	326	211	115	.54
6	311	204	107	.52
7	297	200	97	.49
8	318	207	111	.54
9	342	218	124	.57
Σ	3090	2036	1054	.52

S. C. NICHOLSON
J. JEENEL

Watson Scientific Computing Laboratory
612 West 115th Street
New York 27, New York

1. The IBM-Naval Ordnance Research Calculator, now located at Naval Proving Ground, Dahlgren, Virginia.
2. GEORGE W. REITWIESNER, "An ENIAC determination of π and e to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11–15.
3. For a description of the NORC checking system, see W. J. ECKERT & R. B. JONES, *Faster, Faster*, McGraw-Hill Book Company, New York, 1955, p. 98–104.

Orthogonal Polynomials Arising in the Numerical Evaluation of Inverse Laplace Transforms

Abstract. In finding $f(t)$, the inverse LAPLACE transform of $F(p)$, where (1) $f(t) = (1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^{pt}F(p)dp$, the function $F(p)$ may be either known only numerically or too complicated for evaluating $f(t)$ by CAUCHY's theorem. When $F(p)$ behaves like a polynomial without a constant term, in the variable $1/p$, along $(c - j\infty, c + j\infty)$, one may find $f(t)$ numerically using new quadrature formulas (analogous to those employing the zeros of the LAGUERRE polynomials in the direct Laplace transform). Suitable choice of p_i yields an n-point quadrature

... during electrical discharges and, assuming 100% efficiency of the bond-breaking process in the polythene molecule, has calculated that 3×10^5 discharges are necessary to produce optically detectable damage at any given site. He also quotes the results of experiments, made under alternating-voltage conditions, indicating that 10^9 discharges are necessary at a given site for the start of visible erosion at the inception stress, and deduces that the efficiency of the bond-breaking process need only be low to account for the observed damage. Bearing in mind that the degradation of solid dielectric by electrical discharges is highly stress dependent, the present measurements would appear to be consistent with Garton's figures.

The authors wish to thank M. W. Humphrey Davies for his help in providing the facilities for this work and E. J. Spall for experimental assistance. They are also grateful to the UK Science Research Council for financial support and to British Insulated Callender's Cables Ltd. for the supply of polythene.

S. BEG
B. SALVAGE

Department of Electrical & Electronic Engineering
Queen Mary College
Mile End Road, London E1, England

21st February 1969

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DELTA SERIES.

NUMERICAL INVERSION OF LAPLACE TRANSFORM

An explicit formula for the inversion of the Laplace transform is derived. The formula permits the inverse to be readily evaluated numerically.

A well known problem which arises frequently from the application of the Laplace transform to scientific and engineering problems is the numerical evaluation of the inverse of the transform. The problem has recently received a good deal of attention,[1] and it appears that no completely satisfactory solution has been offered. This letter presents a new method for the numerical inversion of the Laplace transform.

Let $f(t)$ have a Laplace transform

$$F(s) = \int_0^\infty f(t) \exp(-st)dt \qquad \mathrm{Re}(s) > \sigma \quad . \quad . \quad . \quad (1)$$

Thus it is assumed that $f(t)$ is integrable and of exponential order σ.

Let $\delta\left(\frac{\lambda}{t} - 1\right)$ denote the scaled delta function defined by

$$\int_0^T \delta\left(\frac{\lambda}{t} - 1\right) d\lambda = t \qquad 0 < t < T \quad . \quad . \quad . \quad . \quad (2)$$

$$\delta\left(\frac{\lambda}{t} - 1\right) = 0 \qquad t \neq \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$I = \frac{1}{t}\int_0^T f(\lambda)\delta\left(\frac{\lambda}{t} - 1\right)d\lambda \qquad 0 < t < T \quad . \quad . \quad (4)$$

Making use of the property of the delta function given in eqn. 3, whenever t is a point of continuity of f, we can replace the integrand of eqn. 4 by $f(t)\delta\left(\frac{\lambda}{t} - 1\right)$, and therefore

$$I = \frac{f(t)}{t}\int_0^T \delta\left(\frac{\lambda}{t} - 1\right)d\lambda \qquad 0 < t < T \quad . \quad . \quad . \quad (5)$$

Hence, using eqns. 2 and 4, we obtain the sifting integral associated with $\delta\left(\frac{\lambda}{t} - 1\right)$:

$$f(t) = \frac{1}{t}\int_0^T f(\lambda)\delta\left(\frac{\lambda}{t} - 1\right)d\lambda \qquad 0 < t < T \quad . \quad . \quad (6)$$

At those points t where the function f jumps discontinuously from $f(t-)$ to $f(t+)$, the l.h.s. of eqn. 6 should be replaced by

$$\tfrac{1}{2}\{k_1 f(t-) + k_2 f(t+)\}$$

where k_1 and k_2 are real nonnegative constants so that $k_1 + k_2 = 2$. In particular, $k_1 = k_2$ if $\delta\left(\frac{\lambda}{t} - 1\right)$ is defined as the 'limit' of a sequence of functions which are symmetrical about the vertical line $\lambda = t$.

It can be proved (a proof will be presented elsewhere) that the scaled delta function $\delta\left(\frac{\lambda}{t} - 1\right)$ can be expanded into the series

$$\delta\left(\frac{\lambda}{t} - 1\right) = \sum_{i=1}^{\infty} K_i \exp\left(-\alpha_i \frac{\lambda}{t}\right) \quad . \quad . \quad . \quad . \quad (7)$$

More precisely, it can be shown that a sequence of functions $\left\{\delta_N\left(\frac{\lambda}{t} - 1\right)\right\}$ exists, so that at every continuity point t of f,

$$f(t) = \lim_{N\to\infty} f_N(t) \qquad 0 < t < T \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{where } f_N(t) = \frac{1}{t}\int_0^T f(\lambda)\,\delta_N\left(\frac{\lambda}{t} - 1\right) d\lambda \qquad 0 < t < T \quad . \quad (9)$$

$$\delta_N\left(\frac{\lambda}{t} = 1\right) = \sum_{i=1}^{N} K_i \exp\left(-\alpha_i \frac{\lambda}{t}\right) \quad . \quad . \quad . \quad . \quad (10)$$

(*a*) the constants α_i and K_i are either real, or occur in complex-conjugate pairs, e.g. $\alpha_1 = \alpha_2^*$, and hence $K_1 = K_2^*$
(*b*) α_i and K_i depend on N
(*c*) as $N \to \infty$, so also $\mathrm{Re}(\alpha_i) \to \infty$ and $|K_i| \to \infty$
(*d*) $\mathrm{Re}(\alpha_i) > 0$
(*e*) the α_i are distinct, i.e. $\alpha_i = \alpha_j$ if, and only, if $i = j$.

From eqns. 9 and 10, we have

$$f_N(t) = \frac{1}{t}\int_0^T f(\lambda)\sum_{i=1}^{N} K_i \exp\left(-\alpha_i \frac{\lambda}{t}\right) d\lambda \qquad 0 < t < T \,. \quad (11)$$

Hence

$$f_N(t) = \frac{1}{t}\sum_{i=1}^{N} K_i \int_0^T f(\lambda) \exp\left(-\alpha_i \frac{\lambda}{t}\right) d\lambda \qquad 0 < t < T \,. \quad (12)$$

Allowing $T \to \infty$, and using eqn. 1, we obtain

$$f_N(t) = \frac{1}{t}\sum_{i=1}^{N} K_i F(\alpha_i/t) \qquad 0 < t < t_c \quad . \quad . \quad . \quad (13)$$

$$\text{where } t_c = \min_{i=1,2,\ldots N}\{\mathrm{Re}(\alpha_i/\sigma)\} \qquad \sigma \geqslant 0 \quad . \quad . \quad . \quad . \quad (14)$$

As $N \to \infty$, $\mathrm{Re}(\alpha_i) \to \infty$, and hence $t_c \to \infty$. Therefore, using eqn. 8, we obtain the explicit inversion formula

$$f(t) = \lim_{N\to\infty} \frac{1}{t}\sum_{i=1}^{N} K_i F(\alpha_i/t) \qquad 0 < t < \infty \quad . \quad (15)$$

A number of methods for obtaining optimal sets of constants α_i and K_i are being investigated, and the full results will be presented later. One (nonoptimal) set of constants for $N = 10$ is given in Table 1, where $A_i = K_i/\alpha_i$.

Table 1

i	α_i	A_i
1	$5{\cdot}2038 - j15{\cdot}7212$	$-10{\cdot}15471 - j4{\cdot}260437$
2	$5{\cdot}2038 + j15{\cdot}7212$	$-10{\cdot}15471 + j4{\cdot}260437$
3	$8{\cdot}7980 - j11{\cdot}9391$	$189{\cdot}2250 + j250{\cdot}7353$
4	$8{\cdot}7980 + j11{\cdot}9391$	$189{\cdot}2250 - j250{\cdot}7353$
5	$10{\cdot}9343 - j8{\cdot}4096$	$-866{\cdot}2283 - j2313{\cdot}588$
6	$10{\cdot}9343 + j8{\cdot}4096$	$-866{\cdot}2283 + j2313{\cdot}588$
7	$12{\cdot}2261 - j5{\cdot}0127$	$1560{\cdot}540 + j8422{\cdot}502$
8	$12{\cdot}2261 + j5{\cdot}0127$	$1560{\cdot}540 - j8422{\cdot}502$
9	$12{\cdot}8376 - j1{\cdot}666$	$-872{\cdot}8822 - j15431{\cdot}37$
10	$12{\cdot}8376 + j1{\cdot}666$	$-872{\cdot}8822 + j15431{\cdot}37$

The truncated inversion formula of eqn. 13 was used in conjunction with the constants of Table 1 to invert a large number of transforms. One example, the significance of which has been discussed,[2] is given by

$$F(s) = \frac{(s-1)(s-2)(s-20)}{(s-1)(s-2)(s-20)(s+1)} \qquad f(t) = \exp(-t)$$

and the corresponding approximate and exact inverses are shown in Table 2.

Table 2

t	0	0·2	0·4	0·8	1·6	2·2
$f(t)$	1·0000	0·81873	0·67032	0·44933	0·20190	0·11080
$f_{10}(t)$	0·99958	0·81865	0·67051	0·44990	0·20282	0·11181

I am grateful to A. Rodrigues, who computed the results given in the Tables.

V. ZAKIAN

Control Systems Centre
University of Manchester Institute of Science & Technology
PO Box 88, Sackville Street, Manchester 1, England

27th February 1969

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IMPLEMENTING AN ERROR-LOCATING CODE

A binary code which locates the position of a single subblock containing errors in a code word is described briefly, and a decoding technique employing feedback shift registers is discussed.

Although codes for error location have been known for several years,[1] there is little published work on the implementation of these codes. It has been shown that some error-locating codes are equivalent to cyclic codes under co-ordinate permutation,[2] which gives the opportunity of using standard shift-register techniques to determine the syndrome components for a received word.[3] The problem of determining the location of subblocks containing errors from the syndrome remains, however. This letter describes a decoding procedure for an error-locating code which indicates the location of a single subblock containing errors in a code word.

The code used to demonstrate the method is equivalent to the (63, 51) code first described by Wolf and Elspas,[1] and is also a member of a more general class of codes described by Goethals.[2] The 63-digit code word is subdivided into 9 subblocks of 7 digits, and the position of a single subblock containing not more than 6 errors can be located in a code word. A parity-check matrix for the code is formed by the Kronecker product of the parity-check matrices for a (7, 1) cyclic-error-detecting code over $GF(2)$ and a (9, 7) Bose–Chaudhuri–Hocquenghem single-error-correcting code over $GF(2^3)$.

The error-detecting code has a generator polynomial

$$g_1(x) = (1 + x + x^3)(1 + x^2 + x^3)$$

Roots of the generator polynomial are β and β^3, where β is a primitive 7th root of unity.[4] The error-correcting code has a generator polynomial

$$g_2(x) = 1 + \beta^6 x + x^2$$

ω^4 is a root of $g_2(x)$, where ω^4 is a primitive 9th root of unity. The only other root of $g_2(x)$ is ω^5, since $(\omega^4)^{2^3} = \omega^5$ and $(\omega^5)^{2^3} = \omega^4$.

Taking the Kronecker product of the parity-check matrices of these two codes gives the parity-check matrix $\boldsymbol{H}$ for the error-locating code.

$$\boldsymbol{H} = \begin{bmatrix} \omega^0[\beta^0\beta^1\beta^2\beta^3\beta^4\beta^5\beta^6] & \omega^4[\beta^0\beta^1\beta^2\ldots] & \omega^8[\beta^0\beta^1\ldots]\ldots \\ \omega^0[\beta^0\beta^3\beta^6\beta^2\beta^5\beta^1\beta^4] & \omega^4[\beta^0\beta^3\beta^6\ldots] & \omega^8[\beta^0\beta^3\ldots]\ldots \end{bmatrix}$$

This error-locating code can be shown to be equivalent to a cyclic code having generator-polynomial roots $\omega^4\beta = \alpha^{22}$ and $\omega^4\beta^3 = \alpha^{31}$, where α is a primitive 63rd root of unity.[2] Taking the minimum function of α^{22} and α^{31} gives the generator polynomial

$$g_3(x) = (1 + x^5 + x^6)(1 + x^2 + x^3 + x^5 + x^6)$$

The relationship between this code and the double-subblock error-locating code described in Reference 2 will be apparent.

Encoding 51 data digits into the 63-digit equivalent cyclic code word presents no problem, and may be accomplished in the usual manner by using a 12-stage feedback shift register.[3] Digit interchange then takes place, rearranging the positions of the digits to form the error-locating code word.[2] This digit interchange also takes place at the receiver to convert the error-locating received word into its equivalent cyclic form, which enables the syndrome components to be generated using 6-stage feedback shift registers. Four such registers are used, corresponding to substitution of $x = \omega^4\beta$, $\omega^5\beta$, $\omega^4\beta^3$ and $\omega^5\beta^3$ in the received word. Fig. 1 shows the register used for substituting $x = \omega^5\beta$ in the word.

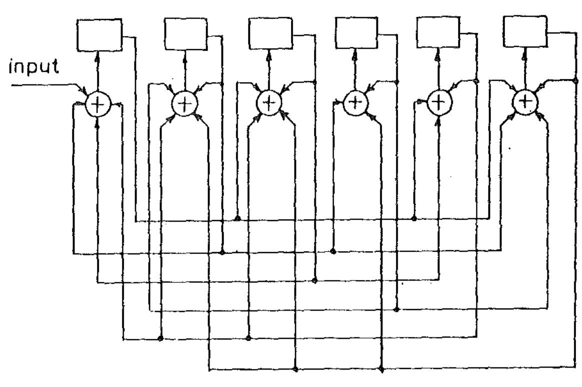

Fig. 1 *Shift register with multiplication by $\omega^4\beta = \alpha^{50}$ on each shift*

Suppose the received word contains a detectable error pattern in subblock j, described by the error vector $v_j(x)$. The four parity-check registers will contain the following syndrome components:

Register 1: $(\omega^4)^j v_j(\beta)$
Register 2: $(\omega^4)^j v_j(\beta^3)$
Register 3: $(\omega^5)^j v_j(\beta)$

Comparing (4) with (2), we get the required formula as

$$b_k = (-1)^k \sum_{i=1}^{C_n^k} \sum_{j=1}^{N_k} (-1)^{L_{ij}^k} C(G_{0i}^k)_j \qquad (k = 1, 2, \cdots, n). \tag{5}$$

The application of this formula is greatly facilitated by means of the graph drawn in Fig. 1, since all the branches are parallel and there are no cross-branches.

As an example let us find the coefficients of the characteristic polynomial

$$P(\lambda) = \lambda^4 - b_1\lambda^3 + b_2\lambda^2 - b_3\lambda + b_4.$$

From the flow graph given in Fig. 1 and formula (5) we have directly

$$\begin{aligned}
b_1 &= -(-a_{11} - a_{22} - a_{33} - a_{44}) \\
&= a_{11} + a_{22} + a_{33} + a_{44}; \\
b_2 &= [a_{33}a_{44}(-1)^2 + a_{34}a_{43}(-1)^1] \\
&\quad + [a_{22}a_{44}(-1)^2 + a_{24}a_{42}(-1)^1] \\
&\quad + [a_{22}a_{33}(-1)^2 + a_{23}a_{32}(-1)^1] \\
&\quad + [a_{11}a_{44}(-1)^2 + a_{14}a_{41}(-1)^1] \\
&\quad + [a_{11}a_{33}(-1)^2 + a_{13}a_{31}(-1)^1] \\
&\quad + [a_{11}a_{22}(-1)^2 + a_{12}a_{21}(-1)^1]; \\
b_3 &= -[a_{22}a_{33}a_{44}(-1)^3 + a_{22}a_{34}a_{43}(-1)^2 \\
&\quad + a_{24}a_{42}a_{33}(-1)^2 + a_{44}a_{23}a_{32}(-1)^2 \\
&\quad + a_{24}a_{32}a_{43}(-1)^1 + a_{23}a_{42}a_{34}(-1)^1] \\
&\quad - [a_{11}a_{33}a_{44}(-1)^3 + a_{11}a_{34}a_{43}(-1)^2 \\
&\quad + a_{13}a_{31}a_{44}(-1)^2 + a_{14}a_{41}a_{33}(-1)^2 \\
&\quad + a_{31}a_{43}a_{14}(-1)^1 + a_{41}a_{34}a_{13}(-1)^1] \\
&\quad - [a_{11}a_{22}a_{44}(-1)^3 + a_{11}a_{24}a_{42}(-1)^2 \\
&\quad + a_{12}a_{21}a_{44}(-1)^2 + a_{14}a_{41}a_{22}(-1)^2 \\
&\quad + a_{21}a_{42}a_{14}(-1)^1 + a_{41}a_{24}a_{12}(-1)^1] \\
&\quad - [a_{11}a_{22}a_{33}(-1)^3 + a_{11}a_{23}a_{32}(-1)^2 \\
&\quad + a_{12}a_{21}a_{33}(-1)^2 + a_{13}a_{31}a_{22}(-1)^2 \\
&\quad + a_{21}a_{32}a_{13}(-1)^1 + a_{31}a_{23}a_{12}(-1)^1]; \\
b_4 &= [a_{11}a_{22}a_{33}a_{44}(-1)^4 + a_{11}a_{22}a_{34}a_{43}(-1)^3 \\
&\quad + a_{11}a_{23}a_{32}a_{44}(-1)^3 + a_{11}a_{24}a_{42}a_{33}(-1)^3 \\
&\quad + a_{12}a_{21}a_{33}a_{44}(-1)^3 + a_{13}a_{31}a_{22}a_{44}(-1)^3 \\
&\quad + a_{14}a_{41}a_{22}a_{33}(-1)^3 + a_{11}a_{42}a_{34}a_{23}(-1)^2 \\
&\quad + a_{11}a_{24}a_{32}a_{43}(-1)^2 + a_{12}a_{21}a_{34}a_{43}(-1)^2 \\
&\quad + a_{31}a_{23}a_{12}a_{44}(-1)^2 + a_{41}a_{24}a_{12}a_{33}(-1)^2 \\
&\quad + a_{13}a_{21}a_{32}a_{44}(-1)^2 + a_{13}a_{31}a_{24}a_{42}(-1)^2 \\
&\quad + a_{13}a_{41}a_{34}a_{22}(-1)^2 + a_{14}a_{21}a_{42}a_{33}(-1)^2 \\
&\quad + a_{14}a_{31}a_{43}a_{22}(-1)^2 + a_{14}a_{41}a_{23}a_{32}(-1)^2 \\
&\quad + a_{12}a_{23}a_{34}a_{41}(-1)^1 + a_{24}a_{12}a_{31}a_{43}(-1)^1 \\
&\quad + a_{13}a_{21}a_{42}a_{34}(-1)^1 + a_{13}a_{41}a_{24}a_{32}(-1)^1 \\
&\quad + a_{14}a_{21}a_{32}a_{43}(-1)^1 + a_{14}a_{31}a_{23}a_{42}(-1)^1].
\end{aligned}$$

MIRKO M. MILIĆ
Electrotech. Faculty
University of Beograd
Beograd, Yugoslavia

$\mathcal{L}$ = RATIO OF RAT. POLYNOMIALS
USES JACOBI POLYNOMIALS

On the Inversion of Laplace Transforms by Means of Truncated Series of Orthonormal Exponential Functions

In his paper "On the Representation of Transients by Series of Orthogonal Functions," Armstrong [1] discusses a procedure for inverting a transfer function $F(s)$ by means of an orthonormal exponential series expansion of $f(t)$. [For the purposes of this paper $F(s)$ is assumed to be of the following form:

$$F(s) = \frac{b_r s^r + b_{r-1}s^{r-1} + \cdots + b_1 s + b_0}{s^q + d_{q-1}s^{q-1} + \cdots + d_1 s + d_0}. \tag{1}$$

The exponentials, as he shows in an earlier paper [2], can be expressed in terms of the Jacobi polynomials $J_n(2,,2|e^{-\beta t})$.[1] Armstrong's inversion formulae are summarized below [(2)[2] – (3c)], since they are referred to throughout the remainder of this article. In addition the series expansion which defines the Jacobi polynomials is given in (4).

$$\mathcal{L}^{-1}\{F(s)\} \approx f_a(t) = \sum_{n=0}^{N} A_n\psi_n(t) \qquad n = 0, 1, \cdots, N \tag{2}$$

$$\begin{cases}
\psi_n(t) = (-1)^n(2\beta)^{1/2}(n+1)^{3/2}e^{-\beta t}J_n(2, 2|e^{-\beta t}) & \text{(3a)} \\
A_n = (-1)^n(2\beta)^{1/2}(n+1)^{3/2}\sum_{m=0}^{n} C_{mn}F[\beta(m+1)] & \text{(3b)} \\
C_{mn} = (-1)^m \dfrac{(m+n+1)!\,n!}{m!\,(n-m)!\,(m+1)!\,(n+1)!} & \text{(3c)}
\end{cases}$$

$$J_n(a, c \mid e^{-\beta t}) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{\Gamma(a+m+n)\Gamma(c)}{\Gamma(a+n)\Gamma(c+m)} e^{-m\beta t}. \tag{4}$$

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[1] This notation for the Jacobi polynomials is in keeping with Armstrong's.

[2] This equation is a truncated version of Armstrong's (2) which is repeated below, for convenience. In practice one truncates the infinite series, (2a), to $N+1$ terms. N can be determined from the truncation error, a measure of which is given by the integral-squared error. The truncated series is designated $f_a(t)$ to distinguish it from $f(t)$. The convergence of $f_a(t)$ is discussed thoroughly in the literature [3] and, therefore, will not be discussed in the present article.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \sum_{n=0}^{\infty} A_n\psi_n(t). \tag{2a}$$

As can be seen, $J_n(a, c|e^{-\beta t})$ is a function of three parameters a, c and the variable t. It is not clear from [1] or [2] why Armstrong fixes the values of a and c at two. It is clear, however, that this choice constrains $f_a(t)$. In particular, one can show from (2) and (3a) that

$$\lim_{s\to\infty} F_a(s) = \lim_{s\to\infty} \Psi_n(s) = \frac{1}{s}, \tag{5}$$

which means, regardless of the high-frequency behavior of $F(s)$, which from (1) is $1/s^{q-r}$, and, which is always known at the onset of the inversion, that $f_a(t)$ will match $f(t)$ poorly at (and in the vicinity of) $t = 0$, for $q - r > 1$. This follows from the Initial-Value Theorem.

Mendel [4] has shown that

$$\phi_n(t) = (-1)^n \sqrt{\frac{\beta}{K_n}}\, e^{-(c\beta/2)t}(1 - e^{-\beta t})^{(a-c)/2} J_n(a, c \mid e^{-\beta t})$$
$$n = 0, 1, \cdots, N \tag{6}$$

where

$$K_n = \frac{n!\,[\Gamma(c)]^2\Gamma(n + a - c + 1)}{(a + 2n)\Gamma(a + n)\Gamma(c + n)} \tag{7}$$

is a set of exponentials which are orthonormal with respect to a uniform weighting function. These exponentials have the property that their Laplace transforms approach $1/s^{[(a-c)/2]+1}$ for large values of s; that is to say,[3]

$$\text{A.O. } (\Phi_n) = \frac{a - c}{2} + 1. \tag{8}$$

The purpose of the present communication is to present a generalization of Armstrong's inversion procedure. Here a and c, in (4), are chosen so as to improve the inverse, which in this case is

$$f_a(t) = \sum_{n=0}^{N} A_n \phi_n(t), \tag{9}$$

for small values of time.

Remark 1. The main advantage of Armstrong's procedure is that it enables one to invert $F(s)$ into a series of exponentials without *a priori* knowledge of the poles of $F(s)$. The inclusion of the asymptotic order of $F(s)$, which is known *a priori*, into the inversion procedure means that the initial values of $f_a(t)$, $f_a^{(1)}(t)$, $\cdots$, and $f_a^{(q-r-2)}(t)$ will equal the zero initial values of $f(t)$, $f^{(1)}(t)$, $\cdots$, and $f^{(q-r-2)}(t)$ respectively; thus, the inverse will be enhanced in the vicinity of zero time, as desired.[4]

Remark 2. It is quite obvious that (3a) is a special case of (6). Specifically,

$$\psi_n(t) = \phi_n(t)\,|_{a=c=2}. \tag{10}$$

The results derived in the following section for A_n in (9) should, therefore, include (3b) as a special case.

Remark 3. The unity asymptotic order case does not necessarily restrict a and c to two; it merely requires $a = c$, as is evident from (8). A procedure for choosing a and c is discussed in the proceeding section.

[3] A. O. (Φ_n) is read "the asymptotic order of $\Phi_n(s)$."

[4] Sets of $\phi_n(t)$ which are orthonormal with respect to nonuniform weighting functions, which are given by an equation similar to (6), and which also are of any asymptotic order are given in Mendel [4]. Their use in the inversion procedure usually complicates the evaluation of A_n considerably; however, their use improves the inverse $f_a(t)$ for, not only small values of time, by virtue of their asymptotic order, but also other intervals of time, by virtue of their normality with respect to a nonuniform weighting function.

Following Armstrong's derivation of A_n, (3b), [1] one can show first, that $\Phi_n(s)$, the Laplace transform of $\phi_n(t)$ in (6), is

$$\Phi_n(s) = (-1)^n \sqrt{\frac{\beta}{K_n}} \sum_{k=0}^{(a-c)/2} \sum_{m=0}^{n} g_{km}^n \frac{1}{s + \left(k + m + \dfrac{c}{2}\right)\beta} \tag{11}$$

where

$$g_{km}^n = (-1)^{k+m} \binom{(a-c)/2}{k} \binom{n}{m} \frac{\Gamma(a + m + n)\Gamma(c)}{\Gamma(a + n)\Gamma(c + m)}; \tag{12}$$

second, that A_n can be evaluated from the integral

$$A_n = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\Phi_n(-s)\, ds \tag{13}$$

by contour integration;[5] and finally, that

$$A_n = (-1)^n \sqrt{\frac{\beta}{K_n}} \sum_{k=0}^{(a-c)/2} \sum_{m=0}^{n} g_{km}^n F\left[\beta\left(k + m + \frac{c}{2}\right)\right]. \tag{14}$$

The complete inversion procedure follows from (9), (6) and (14), once β, a and c are specified.

Remark 4. Note that

$$C_{mn} = g_{km}^n\,|_{a=c=2}. \tag{15}$$

The truth of the conjecture in Remark 2 follows directly.

Remark 5. The specification of β, which corresponds to the spacing of the poles of $\Phi_n(s)$ [4], has been discussed by Armstrong [1] and, therefore, will not be further discussed here. a and c need not be specified arbitrarily. They may, for example, be determined from a specification of A.O. (Φ_n) and α_0[the first pole of $\Phi_n(s)$], as follows:

1) The asymptotic order of $F(s)$ is known; thus, setting A.O. (Φ_n) = A.O. (F), it follows, from (8), that

$$\text{A.O. } (F) = \frac{a - c}{2} + 1. \tag{16}$$

2) The first pole of $\Phi_n(s)$, α_0, is located at

$$\alpha_0 = \frac{c\beta}{2}. \tag{17}$$

Its location is not immediately available from the given information $F(s)$. One procedure for determining α_0 involves an analog computer simulation of $F(s)$, from which $f(t)$ is recorded. Since α_0 is the first pole of $F_a(s)$, it represents the term in $f_a(t)$ with the longest time constant. Suppose, for example, that, from the analog computer simulation of $F(s)$, $f(t)$ is found to approach zero amplitude in 8 seconds. It is safe to assume then, that if a single term in $f(t)$ contributed the 8-second response, it would be of the form $e^{-\frac{1}{2}t}$. Based upon this assumption, one would choose $\alpha_0 = \frac{1}{2}$. a and c are found, from (16) and (17), to be

$$a = 2[\text{A.O. } (F) - 1] + \frac{2\alpha_0}{\beta} \tag{18}$$

$$c = \frac{2\alpha_0}{\beta}. \tag{19}$$

[5] A discussion of the contour integration details can be found in Armstrong [1]; hence, they will not be elaborated upon here.

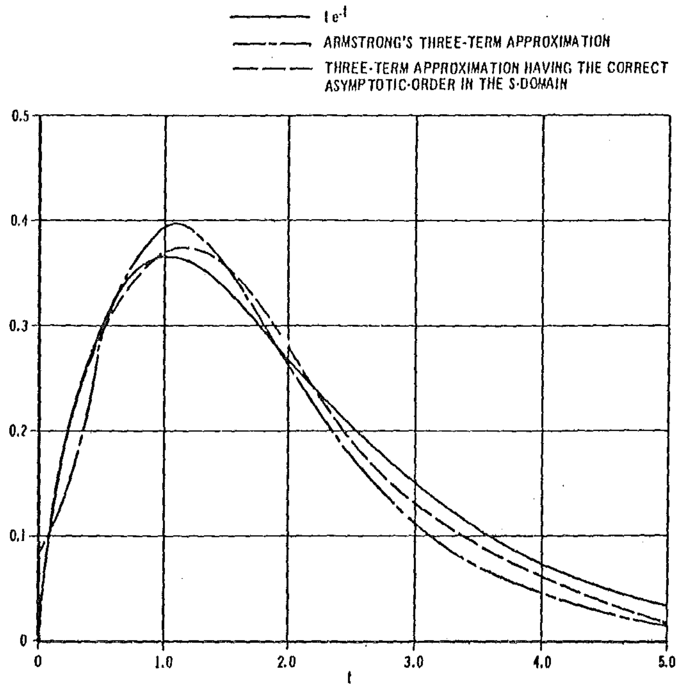

Fig. 1—Plot of Armstrong's and the author's three-term approximations of te^{-t}.

Example

Armstrong [1] considers the inversion of

$$F(s) = \frac{1}{(s+1)^2} \tag{20}$$

by means of the following three-term series:

$$f(t) \approx f_a(t)$$
$$= \sum_{n=0}^{2} (-1)^n A_n \sqrt{2(n+1)^3}\, e^{-t} J_n(2, 2 \mid e^{-t}). \tag{21}$$

The constants A_0, A_1 and A_2 are determined from (3b) with $\beta = 1$. Carrying out these calculations and expanding $f_a(t)$ in (21), it is straightforward to show

$$f_a(t) = 2.5833e^{-t} - 5.0000e^{-2t} + 2.5000e^{-3t}. \tag{22}$$

This function is plotted in Fig. 1 along with $f(t)$. In addition, a three-term result of the inversion procedure, outlined in the preceding section,

$$f_a(t) = 3.0692e^{-t} - 8.5412e^{-2t} + 9.4200e^{-3t} - 3.9480e^{-4t} \tag{23}$$

is also plotted in that figure. The calculations (omitted for the sake of brevity) which were performed in the determination of (23) were based upon the following specifications of β, a and c:

1) The spacing of the poles of $F_a(s)$ in Armstrong's results, (22), is unity; thus, for the sake of comparison, β is also chosen to be unity.
2) A.O. $(F) = 2$; thus, A.O. $(\Phi_n) = 2$.
3) The location of the first pole of $F_a(s)$ in (22) is at $s = -1$. Again for the sake of comparison, α_0 is chosen to be unity.

From 2), 3), and (18) and (19), a and c are found to be 4 and 2 respectively.

It is apparent, from Fig. 1, that both Armstrong's and this author's three-term approximations approximate $f(t)$ quite well [in the sense of closeness-of-fit as measured by the magnitude of the difference between $f(t)$ and $f_a(t)$]. It is also apparent, however, that choosing the correct asymptotic order for the approximation does improve it for small values of time. The price paid for this is the addition of a fourth exponential in the "three-term" approximation (23).[6]

Conclusion

This communication is not intended to be all inclusive. It points out the possibility of and a method for improving the inverse for small values of time. In effect, the inversion procedure discussed above represents one solution to the problem of inverting a transfer function, whose poles are not known *a priori*, such that the initial values of (the inverse) $f_a(t)$, $f_a^{(1)}(t)$, $\cdots$, and $f_a^{(q-r-2)}(t)$ are constrained to the zero initial values of $f(t)$, $f^{(1)}(t)$, $\cdots$, and $f^{(q-r-2)}(t)$ respectively; that is to say, it represents an inversion with "initial-value constraints."

Jerry M. Mendel
Astrodynamics Branch
Missile and Space Systems Div.
Douglas Aircraft Co., Inc.
Santa Monica, Calif.

[6] In general, an $n+1$ term expansion in (9) of asymptotic order λ consists of $n+\lambda$ exponentials.

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Group Delay Characteristics of Chebyshev Filters

Insertion loss characteristics of Chebyshev filters are well known and synthesis procedures are readily available from existing literature.[1] In certain applications, knowledge of the group delay characteri[illegible] is required. Orchard[2] has derived explicit formulas for the g[illegible] delay of both Chebyshev and Butterworth filters and Cohn[3] has presented some group delay curves for $n = 5$. A complete set of curves is however not available in existing literature.

In this communication, normalized group delay characteristics are presented graphically[4] for $n = 2$ through 15 and for ripples of 0.02 db, 0.05 db, 0.1 db, 0.2 db, 0.5 db, and 1.0 db, labelled on the curves as a, b, c, d, e, and f, respectively. Two graphs are drawn for each n to give a clear presentation. Since the peaks of the curves lie beyond $\omega = 1.2$ for $n = 2$, 3, and 4 with ripples a, b, and c, the frequency range has been extended to 2.4 to show up the peaks. For $n = 13$, 14 and 15 with ripples d, e, and f, an expansion of the frequency scale between 0.9 and 1.05 has been found necessary. The vertical scale in each graph is the group delay normalized to the dc value. These values are shown in Table I. The curves are applicable to both the zero insertion loss response

$$|t(j\omega)|^2 = \frac{1}{1 + h^2 Tn^2(\omega)}$$

and the finite insertion loss functions

$$|t(j\omega)|^2 = \frac{1}{1 + k^2 + h^2 Tn^2(\omega)}$$

since in the former case the ripple is given by $1 + h^2$ and in the latter by $1 + k^2 + h^2/1 + k^2$. The poles are identical if the ripples are the same so that the phase responses will be the same in this case.

TABLE I

n	RIPPLE—DB 0.02	0.05	0.10	0.20	0.50	1.00
2	0.503	0.618	0.716	0.818	0.940	0.996
3	1.243	1.436	1.605	1.804	2.145	2.521
4	2.087	2.292	2.445	2.583	2.705	2.694
5	3.029	3.287	3.506	3.759	4.206	4.726
6	3.957	4.190	4.354	4.488	4.563	4.456
7	4.956	5.243	5.487	5.776	6.306	6.958
8	5.901	6.142	6.303	6.423	6.439	6.231
9	6.925	7.230	7.492	7.810	8.417	9.197
10	7.871	8.113	8.268	8.369	8.322	8.010
11	8.911	9.229	9.506	9.849	10.53	11.44
12	9.852	10.09	10.24	10.32	10.21	9.791
13	10.90	11.23	11.52	11.89	12.65	13.68
14	11.84	12.07	12.21	12.27	12.09	11.57
15	12.90	13.24	13.54	13.94	14.76	15.92

Note: Curves for $n = 2 - 15$ are shown in Figs. 1–14, on pages 105–108.

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JIN-TWAN LIM
J. O. SCANLAN
Dept. of Elec. Engrg.
University of Leeds,
Leeds, Yorkshire, England

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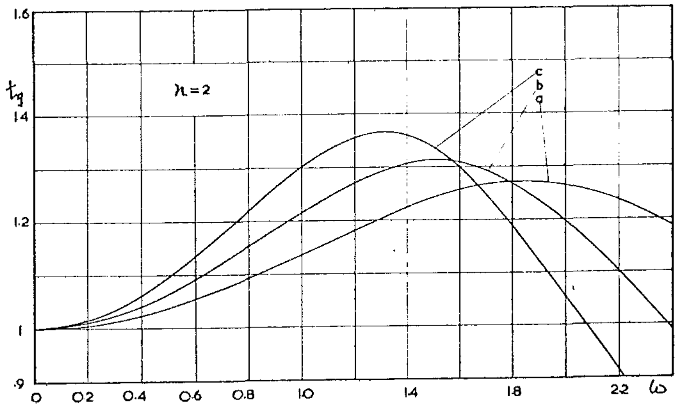

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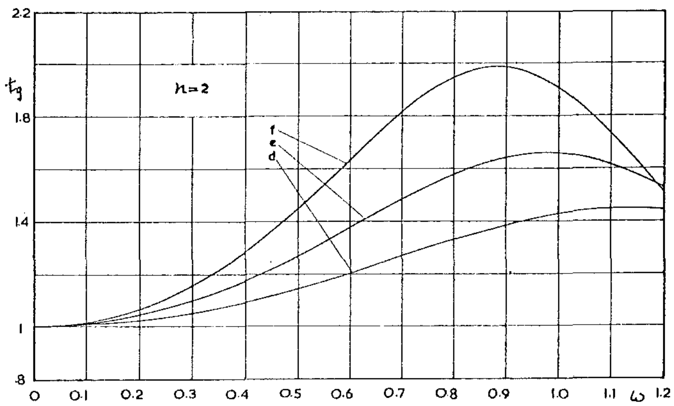

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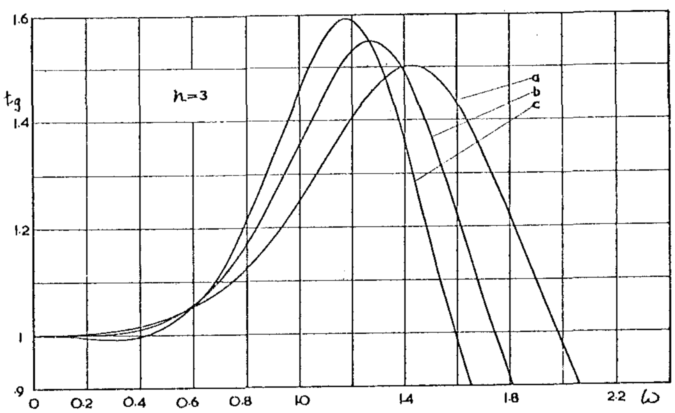
Fig. 1.



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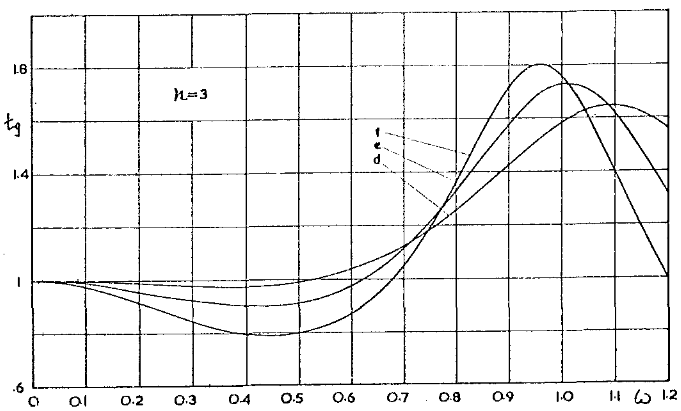

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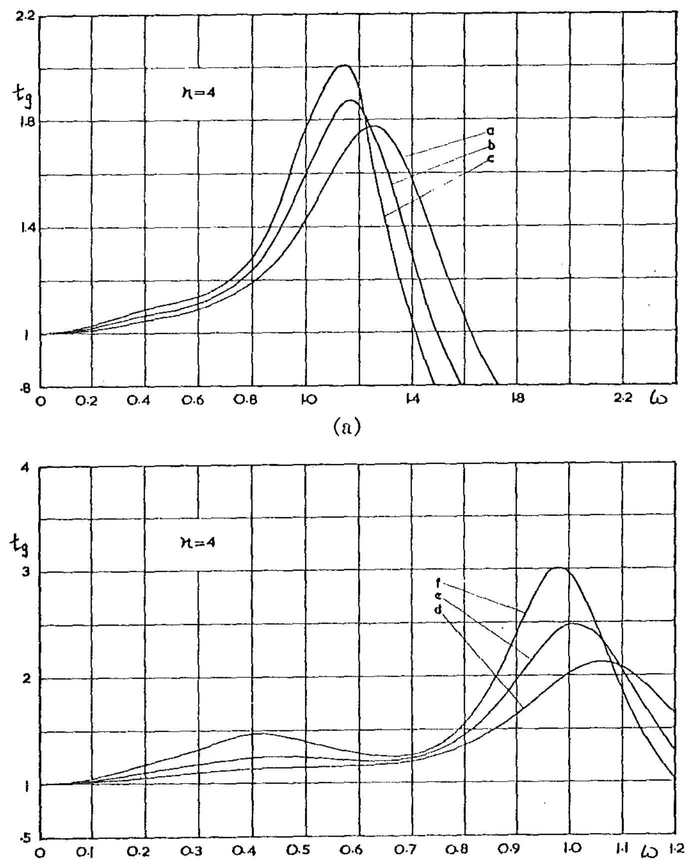
Fig. 2.



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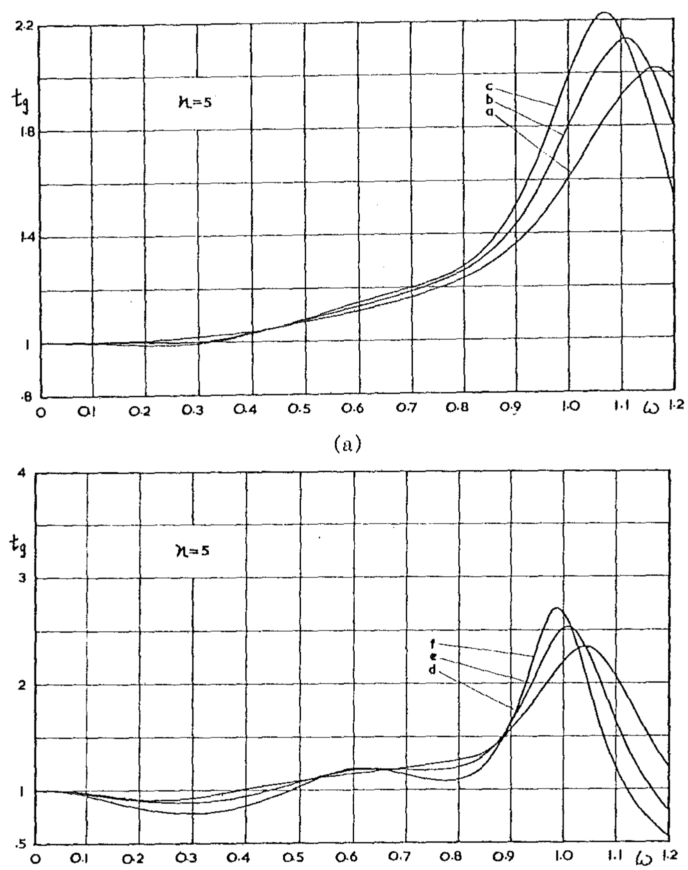
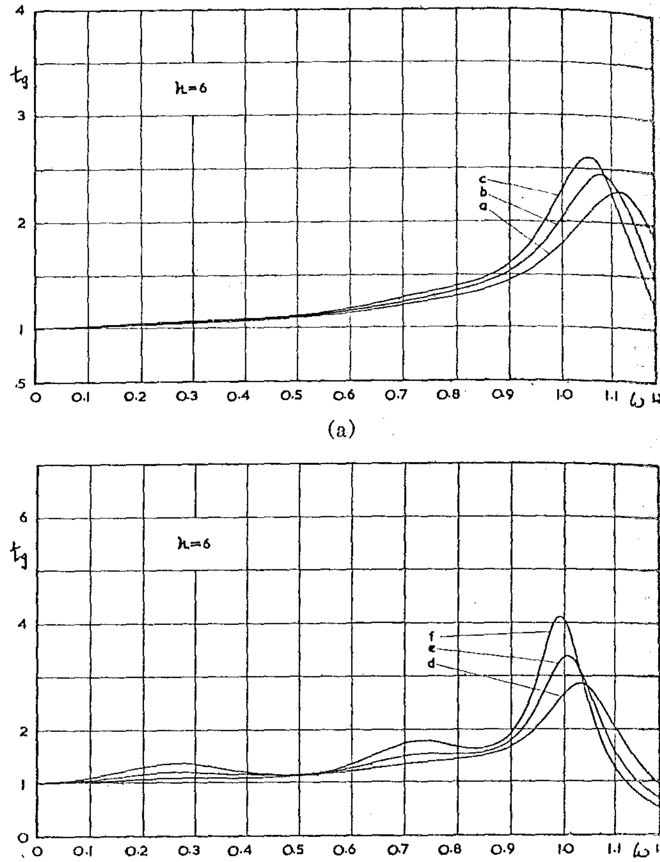


Fig. 4.



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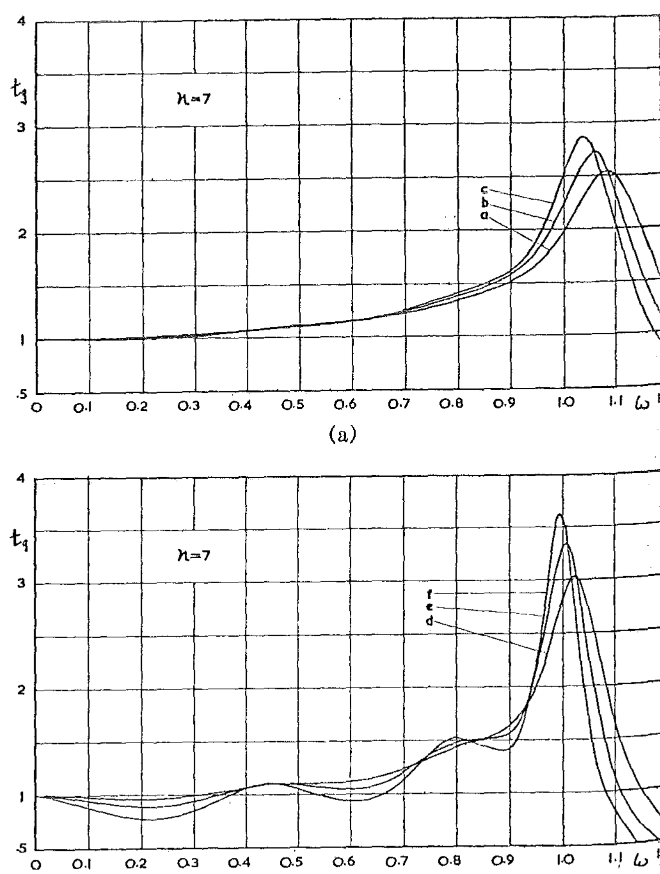
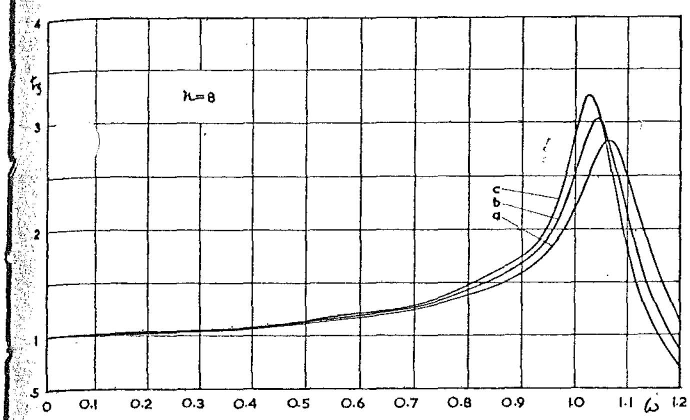


Fig. 6.



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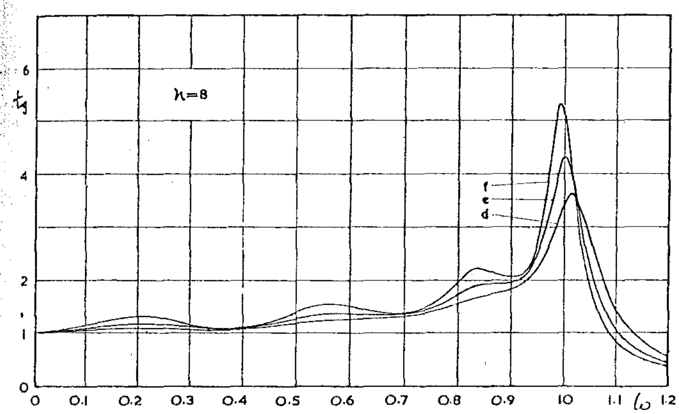

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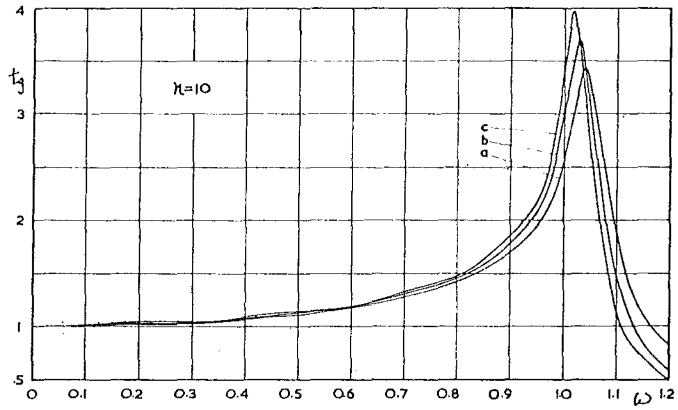
Fig. 7.



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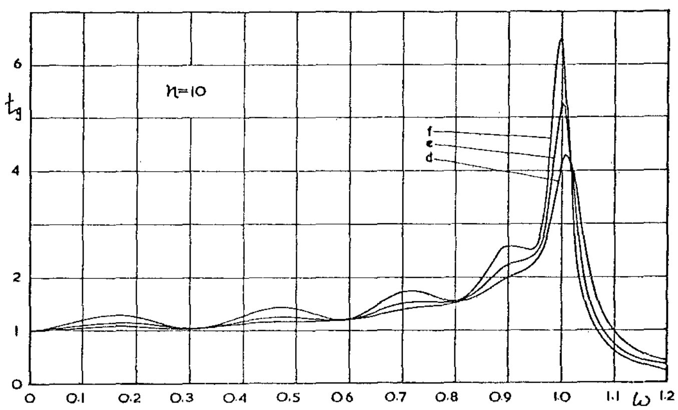

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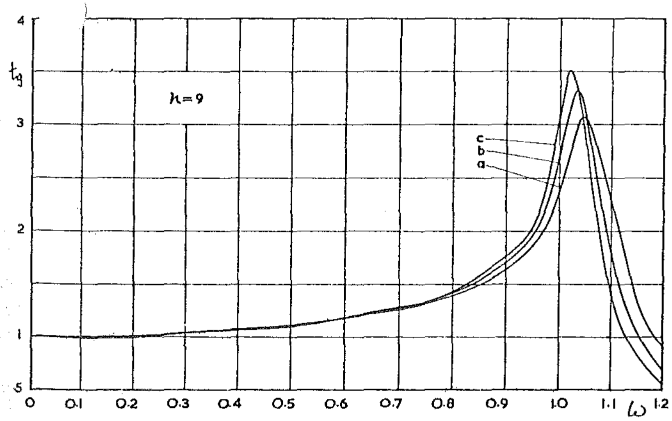
Fig. 9.



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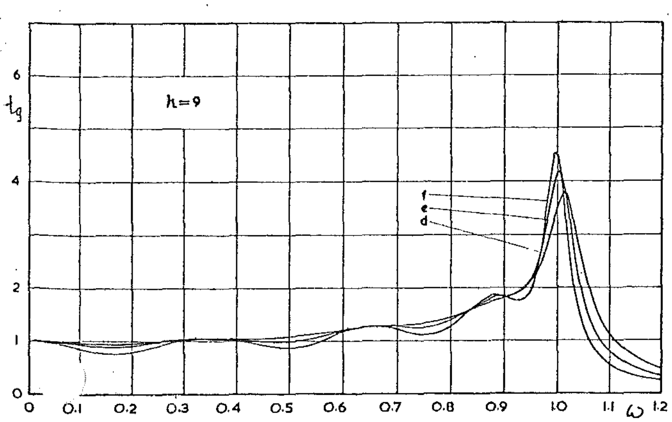

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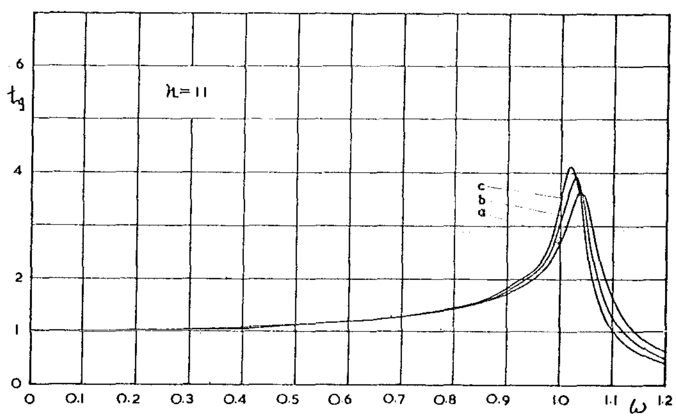
Fig. 8.



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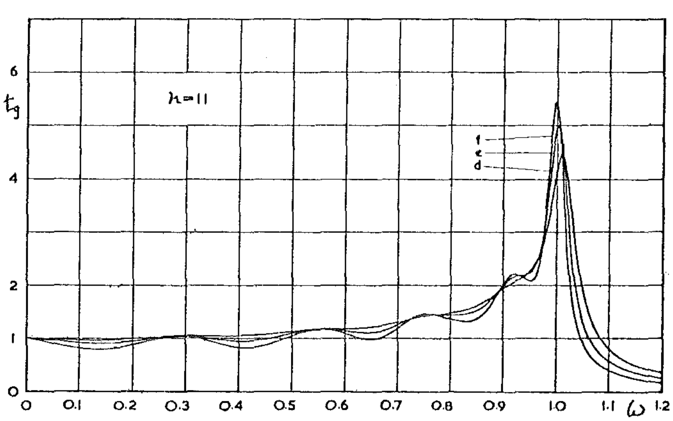

(b)

Fig. 10.

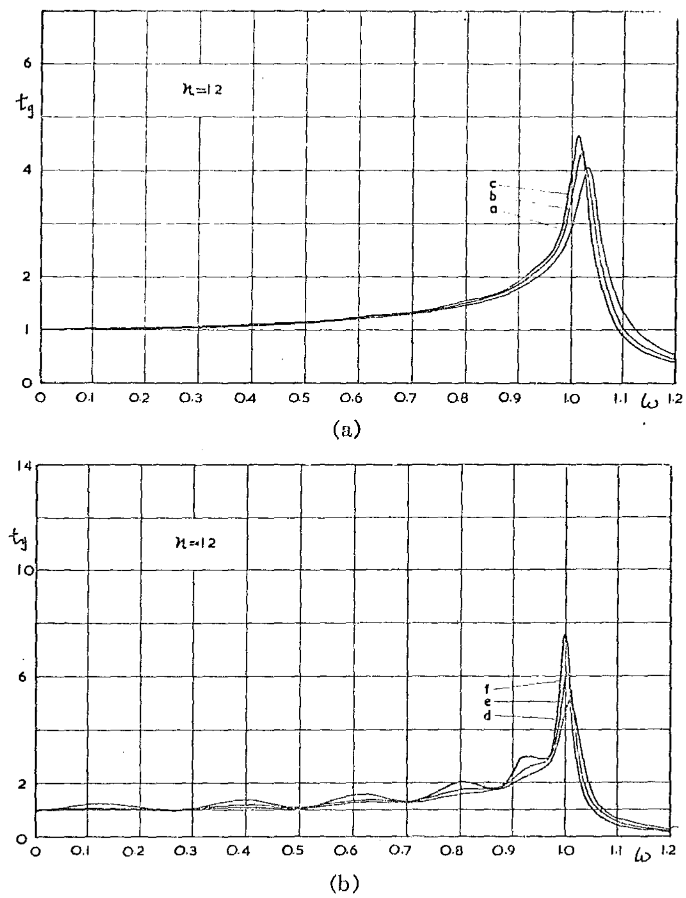

Fig. 11

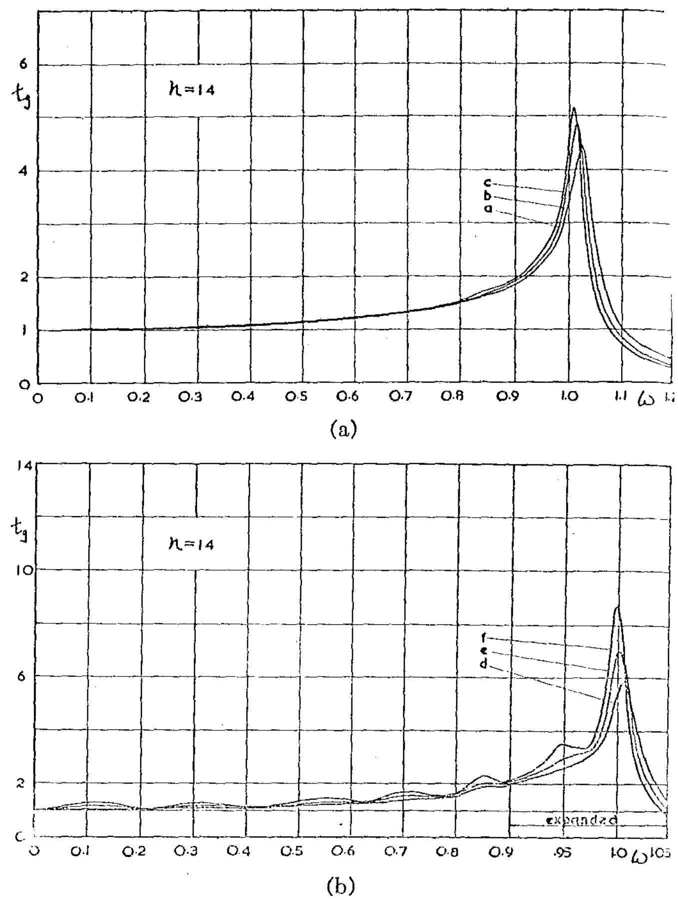

Fig. 13.

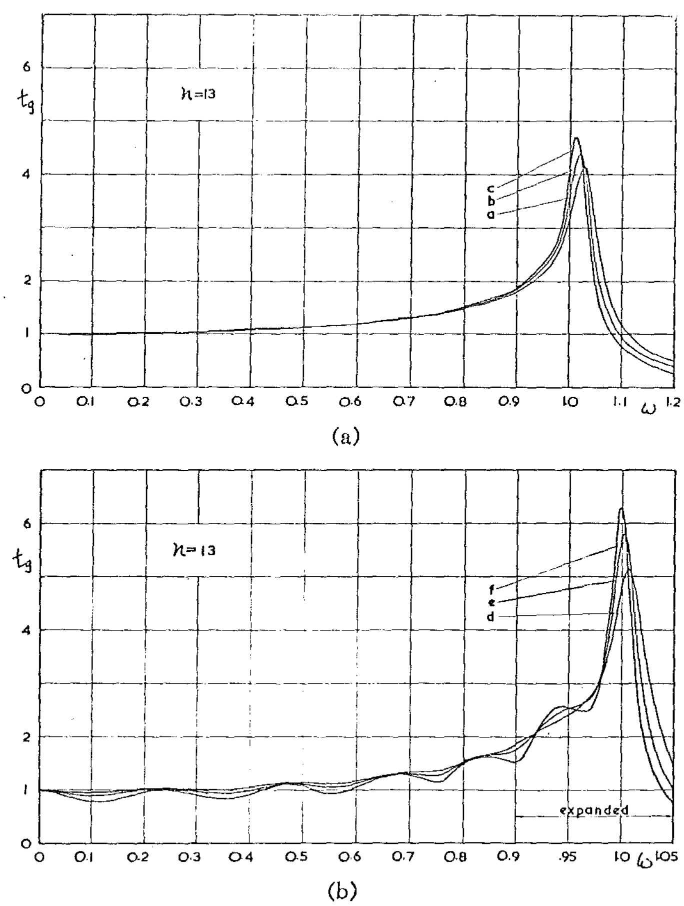

Fig. 12.

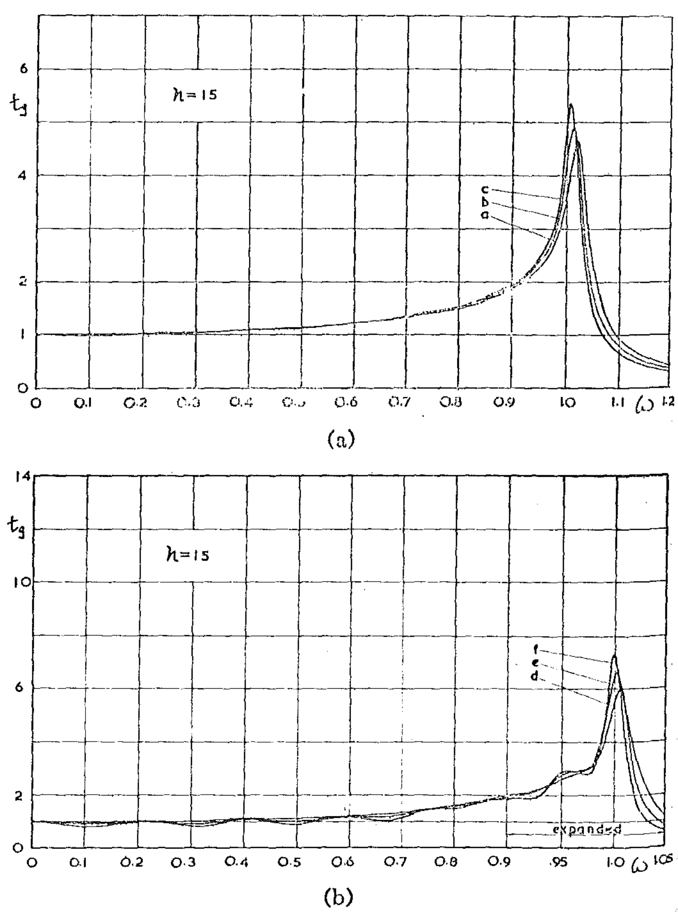

Fig. 14.

VOL. 60 #11

probability of error P_e, and hence may be proposed as a suitable criterion for the selection of effective patterns in multiclass pattern recognition.

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C. Chitti Babu
School of Eng.
Univ. California
Irvine, Calif. 92664

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SADDLE POINT INTEGRATION

Numerical Calculation of Cumulative Probability from the Moment-Generating Function

Abstract—A numerical method for determining the cumulative probability distribution of a nonnegative random variable is based on the steepest descent approximation of the inverse Laplace transform of its moment-generating function. Good numerical agreement with the cumulative exponential and Poisson distributions is demonstrated.

It is often important in detection theory to late the tail distribution of a nonneg e random variable g, that is, the probability that g exceeds a certain value g_0, or

$$Q(g_0) = \Pr[g > g_0] = \int_{g_0}^{\infty} P(g)dg. \quad (1)$$

In many problems the Laplace transform of the probability density function (PDF) $P(g)$, defined by

$$h(s) = E[e^{-sg}] = \int_0^{\infty} e^{-sg}P(g)dg \quad (2)$$

where $s=\alpha+i\omega$ is complex with $\alpha>0$, is easily determined.

In recent letters, Helstrom[1] and Nuttall[2] presented numerical techniques for calculating the cumulative probability from $h(s)$. However, neither technique gave good accuracy for the tail distribution. On the other hand, for a nonnegative random variable, the tail distribution can be approximated by the asymptotic expansion of the inverse transform of $h(s)$ through the steepest descent method. The numerical calculation is not complicated, especially when there is only one saddle point involved. Daniels[3] discussed the approximation of the PDF of the sample-mean statistic. Rice[4] applied the steepest descent method to approximate the cumulative distribution of the noncentral chi-square statistic and presented a more general discussion for cases involving more than one saddle point. In the present letter, only one saddle point is considered.

The tail distribution from (1) can be expressed in terms of the Laplace transform $h(s)$ by

$$Q(g_0) = 1 - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s} e^{sg_0}h(s)ds$$
$$= 1 - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s} e^{g_0\phi(s)}ds \quad (3)$$

where

$$\phi(s) = g_0^{-1}\ln h(s) + s \quad (4)$$

is the complex phase of the integral. Assume the integrand has only one single real saddle point, which can be determined from the equation $(d/ds)\phi(s)=0$,

$$g_0 = \frac{d}{ds}\ln h(s)\,\Big|_{s=s_0}. \quad (5)$$

Then the integral in (3) can be approximated by a uniform asymptotic expansion[4] in terms of g_0 and the derivatives of the complex phase evaluated at s_0:

$$Q(g_0) = 1 - E(g_0) - I(g_0) \quad (6)$$

where

$$E(g_0) = \begin{cases} 1 - \operatorname{erfc}[(-2g_0\phi(s_0))^{1/2}], & s_0 < 0 \\ \operatorname{erfc}[(-2g_0\phi(s_0))^{1/2}], & s_0 > 0 \end{cases}$$

and

$$I(g_0) = \frac{\exp[g_0\phi(s_0)]}{[2\pi g_0\phi^{(2)}(s_0)]^{1/2}} \cdot \sum_{k=0}^{\infty}\Bigg\{\left(\frac{-2}{g_0\phi^{(2)}(s_0)}\right)^k s_0^{-1} \cdot \sum_{n=0}^{2k}(-s_0)^{-2k+n}\sum_{l=0}^{n}A_{l,n}(\tfrac{1}{2})_{l+k} - \operatorname{sgn}(s_0)(\tfrac{1}{2})_k\left(\frac{-\phi^{(2)}(s_0)}{2\phi(s_0)}\right)^{1/2} \cdot (g_0\phi(s_0))^{-k}\Bigg\}$$

where

$$\operatorname{erfc} y = \frac{1}{\sqrt{2\pi}}\int_y^{\infty}\exp(-\alpha^2/2)\,d\alpha$$

$$\operatorname{sgn}(s_0) = 1, \qquad \text{for } s_0 > 0$$

$$\operatorname{sgn}(s_0) = -1, \qquad \text{for } s_0 \le 0$$

$$A_{l,n} = \begin{cases} 0, & \text{for } n < l \text{ or } l = 0,\ n \ge 1 \\ 1, & \text{for } l = n = 0 \end{cases}$$

$$A_{l+1,n+1} = \frac{-2}{n+1}\sum_{m=1}^{n-l+1}\frac{m\phi^{(m+2)}(s_0)}{(m+2)!\phi^{(2)}(s_0)}A_{l,n-m+1}$$

$$(\tfrac{1}{2})_m = (\tfrac{1}{2})(\tfrac{1}{2}+1)\cdots(\tfrac{1}{2}+m-1),\ (\tfrac{1}{2})_0 = 1$$

and

$$\phi^{(n)}(s) = g_0^{-1}\left(\frac{d}{ds}\right)^n \ln h(s), \qquad \text{for } n \ge 2.$$

The coefficients $A_{l,n}$ can be obtained by the recurrence relation through the derivatives of the complex phase $\phi(s)$ at $s=s_0$. This scheme is easily programmed for a digital computer. The tail distribution for $g>g_0$ is then obtained by adding up the terms in the asymptotic expansion given by (6) until they become insignificantly small or until they stop decreasing and begin to increase.

When the random variable g is Gaussian distributed, the term $I(g_0)$ in (6) vanishes, and the asymptotic expansion provides the exact tail distribution, which is the error-function integral. Equation (5) shows that the value g_0 at $s_0=0$ is the mean value of the random variable g. The expansion from (6) will diverge at $s_0=0$ because the origin is also a simple pole of the integrand. The tail distribution at this particular point can be approximated by interpolation from neighboring points, or other methods[1,2] can be used.

For a discrete random variable g, the tail distribution is

$$Q(g_0) = \Pr[g > g_0] = \sum_{g>g_0}^{\infty} P(g) \quad (7)$$

and the moment generating function is given by

$$h_d(s) = E[e^{-sg}] = \sum_{g=0}^{\infty} P(g)e^{-sg}. \quad (8)$$

The calculation of the tail distribution is simpler and more accurate if one first approximates the probability $p(g)$ and then adds up the probabilities for all $g>g_0$ as given by (7). For instance, when g takes only nonnegative integral values

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TABLE I
EXPONENTIAL DISTRIBUTION

g_0	0.05	0.3	0.6	0.7	2.5	10	20	40	60
Error %	0.0072	0.0194	0.3243	0.3310	−0.0174	−0.0866	−0.0866	0.2039	0.6614

TABLE II
POISSON DISTRIBUTION

g_0	6	10	14	18	22	26	30	35
Error %	−0.0061	−0.0062	−0.0087	−0.0049	−0.0043	−0.0058	−0.0045	−0.0028

$$P(g) = \frac{1}{2\pi i}\int_{\alpha'-i\pi}^{\alpha'+i\pi} \exp\,[g\phi_d(s)]ds$$
$$\sim \frac{\exp\,[g\phi_d(s_0)]}{[2\pi g\phi_d^{(2)}(s_0)]^{1/2}} \sum_{m=0}^{\infty} \left(\frac{-2}{g\phi_d^{(2)}(s_0)}\right)^m \cdot \sum_{l=0}^{2m} A_{l,2m}(\tfrac{1}{2})_{l+m} \qquad (9)$$

with

$$\phi_d(s) = g^{-1}\ln h_d(s) + s \qquad (10)$$

where the integral is over an interval of length 2π with $\alpha'>0$. The saddle point s_0, the coefficients $A_{l,2m}$, and the derivatives of the complex phase $\phi_d(s)$ can be obtained as before.

Examples

1) Exponential distribution

$$P(g) = \begin{cases} \exp(-g), & g \geq 0 \\ 0, & g < 0 \end{cases} \qquad (11)$$

where

$$h(s) = (1+s)^{-1}$$
$$\phi(s) = -g_0^{-1}\ln(1+s) + s$$
$$s_0 = (1-g_0)/g_0$$
$$\phi^{(n)}(s_0) = (-1)^n g_0^{n-1}(n-1)!, \qquad \text{for } n \geq 2.$$

The tail distribution calculated by the asymptotic expansion from (6) is compared with the exact value in Table I, which shows the percentage errors for several values of g_0.

2) Poisson distribution

$$P(g) = e^{-\lambda}\lambda^g/g! \qquad (12)$$

where

$$h_d(s) = \exp\,[\lambda(e^{-s}-1)]$$
$$\phi_d(s) = g^{-1}[\lambda(e^{-s}-1)] + s$$
$$s_0 = \ln(\lambda/g)$$
$$\phi_d^{(n)}(s_0) = (-1)^n, \qquad \text{for } n \geq 2.$$

The numerical calculation of the tail distribution by (9) and (7) is compared with the exact value and the percentage errors for different g_0 are shown in Table II for $\lambda = 15$.

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LILY WANG
Dep. Appl. Phys. and Inform. Sci.
Univ. California, San Diego
La Jolla, Calif. 92037

Noise in Two-Way Cable-Communications Systems

Abstract—**Equations for determining the reverse direction noise are provided. This noise results in a reduction of the system signal-to-noise ratio in the forward direction. The resulting correction term describes the reduction and is a function of the type and number of vertices in the system tree.**

Consider the cable-communications system representation [1] in Fig. 1. The vertex [2] designated by 1 is a signal source, or central vertex, for the other vertices in the graph. This source provides transmission in the forward direction. Vertices 2 and 3 are designated as terminal vertices. Additionally, cable length l_3 contains m_3 vertices where amplifiers provide gain and equalization [3] for signals en route to the central vertex. Each of these reverse direction amplifiers has gain g_3, and similar statements apply to the other cable lengths. Moreover, unity gain exists for cable and amplifier pairs.

The noise returning to the central vertex is identified as either amplified noise (N_a) or network noise (N_n). This description results from the following noise figure equation [4] for cascaded networks

$$f_{123} = f_1 + (f_2-1)/g_1 + (f_3-1)/g_1g_2 \quad (1)$$

and is rewritten as

$$\begin{aligned} f_{123}g_1g_2g_3KTB = {} & (KTB)g_1g_2g_3 \\ & + [(f_1-1)KTBg_1]g_2g_3 \\ & + [(f_2-1)KTBg_2]g_3 \\ & + [(f_3-1)KTBg_3]. \end{aligned} \qquad (2)$$

This output noise equation contains a source noise term KTB amplified by the product of gains $g_1g_2g_3$. The other terms represent the noise originating from within the networks. Specifically, the noise originating in network 1 is $(f_1-1)KTBg_1$, and it is amplified by an amount g_2g_3.

Returning to Fig. 1, the noise at the central vertex from terminating vertices 2 and 3 is equal to $2KTB$ with unity gain existing for cable and amplifier pairs. For p terminating vertices this noise contribution becomes

$$N_a = pKTB. \qquad (3)$$

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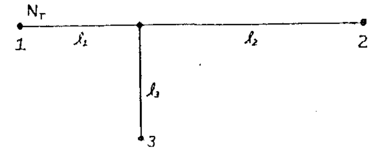

Fig. 1. Two-way system noise example.

For cable length l_i, there are m_i amplifiers and each has a noise figure f_i. The resulting network noise (N_n) is $\sum_{i=1}^{3}(m_if_i-1)KTB$, and the total noise in this example is

$$N_T = 2KTB + \sum_{i=1}^{3}(m_if_i-1)KTB. \quad (4)$$

With M cable lengths and p terminating vertices this equation becomes

$$N_T = pKTB + \sum_{i=1}^{M}(m_if_i-1)KTB. \quad (5)$$

To include additional noise contributions from specific vertices, the noise term N_{d_i} is added. Thus the returning noise becomes

$$N_T = pKTB + \sum_{i=1}^{M}((m_if_i-1)KTB + N_{d_i}). \quad (6)$$

The total noise after m amplifiers in the forward direction, as a result of noise addition from the reverse direction, is

$$N_F = mgfKTB + N_T. \qquad (7)$$

Equation (7) is next converted to dBmV (0 dBmV corresponds to 1 mV) by using logarithms and subtracting the result from the amplifier output signal value S in dBmV. Symbols G and F are for the power gain g and noise figure f, respectively. These quantities are expressed in decibels. Accordingly, the resulting signal-to-noise ratio is

$$\begin{aligned} 10\log(s/N_F) = {} & S - G - F - 10\log KTB \\ & - 10\log m - C \end{aligned} \quad (8)$$

where the correction term C is given by

$$C = 10\log(1 + N_T/(mgfKTB)). \qquad (9)$$

This signal-to-noise ratio equation without the correction term has appeared in [3].

These equations provide a simple means of determining the signal-to-noise ratios in systems with partial or complete two-way usage.

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J. R. HARRER
James R. Harrer Consultant
Costa Mesa, Calif. 92626

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A NEW METHOD OF INVERSION OF THE LAPLACE TRANSFORM*

BY

ATHANASIOS PAPOULIS

Polytechnic Institute of Brooklyn

Introduction. In determining a function $r(t)$ from its Laplace transform $R(p)$

$$R(p) = \int_0^\infty e^{-pt} r(t)\, dt \tag{1}$$

one applies either a partial fraction expansion or an integration along some contour in the complex p-plane; one thus obtains $r(t)$ in terms of the poles and residues of $R(p)$, or from the values of $R(p)$ on a contour of the p-plane. Both methods have obvious disadvantages for a numerical analysis.

In the following we propose to develop a method for determining $r(t)$ in terms of the values of $R(p)$ on an infinite sequence of equidistant points

$$p_k = a + k\sigma \qquad k = 0, 1, \cdots, n, \cdots \tag{2}$$

on the real p-axis, where a is a real number in the region of existence of $R(p)$, and an arbitrary positive integer. That $R(p)$ is uniquely determined from its values at the above points, is known [1]. It should therefore be possible to express $r(t)$ directly in terms of $R(a + k\sigma)$. In this paper it will be shown that $r(t)$ can be written in the form

$$r(t) = \sum_{k=0}^{\infty} C_k \varphi_k(t), \tag{3}$$

where the φ_k's are known functions, and the constants C_k can readily be determined from the values of $R(p)$ at the points $a + k\sigma$.

The φ_k's can be chosen from several sets of complete orthogonal functions; in our discussion we shall use the familiar trigonometric set, the Legendre set and the Laguerre polynomials.

The trigonometric set. We introduce the variable θ defined by

$$e^{-\sigma t} = \cos\theta \qquad \sigma > 0. \tag{4}$$

The $(0, \infty)$ interval transforms into the interval $(0, \pi/2)$, and $r(t)$ becomes

$$r\left(-\frac{1}{\sigma}\ln\cos\theta\right).$$

For simplicity of notation we shall denote the above function by $r(\theta)$ using the same letter r.

The defining equation (1) takes the form

$$\sigma R(p) = \int_0^{\pi/2} (\cos\theta)^{(p/\sigma)-1} \sin\theta\, r(\theta)\, d\theta \tag{5}$$

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hence with

$$p = (2k + 1)\sigma \qquad k = 0, 1, 2, \cdots$$

we have

$$\sigma R[(2k + 1)\sigma] = \int_0^{\pi/2} (\cos\theta)^{2k} \sin\theta\, r(\theta)\, d\theta. \tag{6}$$

In the following we shall assume, without loss of generality, that $r(0) = 0$ subtracting, if necessary, a constant from $r(\theta)$. The function $r(\theta)$ can be expanded in the $(0, \pi/2)$ interval into an odd-sine series

$$r(\theta) = \sum_{k=0}^{\infty} C_k \sin(2k + 1)\theta. \tag{7}$$

This can of course be done by properly extending the definition of $r(\theta)$ in the $(-\pi, +\pi)$ interval.

We shall next determine the coefficients C_k. We have

$$(\cos\theta)^{2n} \sin\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^{2n} \frac{e^{i\theta} - e^{-i\theta}}{2j},$$

expanding in the right hand side and properly collecting terms we obtain

$$2^{2n}(\cos\theta)^{2n} \sin\theta = \sin(2n + 1)\theta + \cdots$$

$$+ \left[\binom{2n}{k} - \binom{2n}{k-1}\right] \sin[2(n - k) + 1]\theta + \cdots + \left[\binom{2n}{n} - \binom{2n}{n-1}\right] \sin\theta. \tag{8}$$

We next insert (7) and (8) into (6); because of the orthogonality of the odd sines in the $(0, \pi/2)$ interval and since

$$\int_0^{\pi/2} [\sin(2n + 1)\theta]^2\, d\theta = \frac{\pi}{4},$$

we have

$$\sigma R[(2n + 1)\sigma] = 2^{-2n} \frac{\pi}{4} \left\{\left[\binom{2n}{n} - \binom{2n}{n-1}\right] C_0 + \cdots + \left[\binom{2n}{k} - \binom{2n}{k-1}\right] C_{n-k} + \cdots + C_n\right\}$$

hence with $n = 0, 1, 2, \cdots$ we obtain the system

$$\frac{4}{\pi}\sigma R(\sigma) = C_0,$$

$$2^2 \frac{4}{\pi}\sigma R(3\sigma) = C_0 + C_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2^{2n} \frac{4}{\pi}\sigma R[(2n + 1)\sigma] = \left[\binom{2n}{n} - \binom{2n}{n-1}\right] C_0 + \cdots + \left[\binom{2n}{k} - \binom{2n}{k-1}\right] C_{n-k} + \cdots + C_n. \tag{9}$$

Thus $R(\sigma)$ gives C_0, $R(3\sigma)$ give C_1 and each value of $R(p)$ at the points $(2k + 1)\sigma$ together with the coefficients $C_0, C_1 \cdots, C_{k-1}$, determines C_k. The system (9) can obviously be written in such a way as to give directly C_k in terms of $R(\sigma)$, $R(3\sigma)$, $\cdots$ alone, but not much is gained, since in a numerical evaluation of the C_k's equation (9) can be used as easily. Table 1 gives the numerical values of the coefficients of the C_k's in the right hand side of (9), for $k = 0, 1, \cdots, 10$.

TABLE 1

n	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
0	1										
1	1	1									
2	2	3	1								
3	5	9	5	1							
4	19	28	20	7	1						
5	42	90	75	35	9	1					
6	132	297	275	154	54	11	1				
7	429	1001	1001	637	273	77	13	1			
8	1430	3432	3640	2548	1260	440	104	15	1		
9	4862	11934	13260	9996	5508	2244	663	135	17	1	
10	16796	41990	48450	38760	23256	10659	3705	950	170	19	1

Thus a method of analysis has resulted which compares sometimes favorably with the known methods of numerical evaluation of $r(t)$. Indeed the computation of $R((2k + 1)\sigma)$ presents no difficulty, and the C_k's can be readily determined from (9); the trigonometric functions are available, hence $r(\theta)$ can be computed with any desired accuracy from the series (7). In a numerical evaluation of $r(\theta)$ one computes the finite sum

$$r_N(\theta) = \sum_{k=0}^{N} C_k \sin (2k + 1)\theta \tag{10}$$

of the first $N + 1$ terms of (7); as N tends to infinity $r_N(\theta)$ tends to $r(\theta)$. The nature of the approximation is well known from the theory of Fourier series [2]; $r_N(\theta)$ and $r(\theta)$ are related by the equation

$$r_N(\theta) = \frac{4}{\pi} \int_0^{\pi/2} r(y) \frac{\sin [\frac{1}{2}(4N + 3)(\theta - y)]}{\sin \frac{1}{2}(\theta - y)} dy, \tag{11}$$

thus the approximating function $r_N(\theta)$ is the average of $r(\theta)$ with the Fourier kernel

$$\frac{\sin [\frac{1}{2}(4N + 3)(\theta - y)]}{\sin \frac{1}{2}(\theta - y)}$$

as the weighting factor. From $r(\theta)$ one can readily obtain $r(t)$ with the change of variable established by (4); however, Eq. (7) can be written directly in the time domain. Indeed since

$$\frac{\sin n\theta}{\sin \theta} = U_n(x) \qquad \cos \theta = x,$$

where $Un(x)$ are the Tchebycheff sine-polynomials of order n and

$$\sin \theta = (1 - e^{-2\sigma t})^{1/2}$$

we have from (7)

$$r(t) = (1 - e^{-2\sigma t})^{1/2} \sum_{k=0}^{\infty} C_k U_{2k}(e^{-\sigma t}). \tag{12}$$

The choice of σ depends on the interval $(0, T)$ in which $r(t)$ is best to be described; if it is chosen so that

$$e^{-\sigma T} = \tfrac{1}{2}$$

then the $(0, T)$ interval transforms into the $(0, \pi/3)$ interval. If a detailed description of $r(t)$ is desired both near the origin and for large values of t, then the function can be evaluated twice with two different values of σ.

The above provides a simple proof of the announced theorem that the Laplace transform $R(p)$ is uniquely determined from its values at the sequence

$$p_k = a + k\sigma \qquad k = 0, 1, \cdots, n, \cdots \tag{2}$$

of equidistant points on the real p-axis. This proof uses the well-known orthogonality and completeness of the trigonometric set. Indeed $r(\theta)$, and hence $r(t)$, is completely determined from the coefficients C_k of (7); these coefficients can be determined from $R(a + k\sigma)$; knowing $r(t)$ one clearly has $R(p)$ therefore $R(p)$ is uniquely determined from its values at the points (2).

The Legendre set. We shall next expand $r(t)$ into a series of Legendre polynomials. We introduce the logarithmic time-scale x defined by

$$e^{-\sigma t} = x \qquad \sigma > 0. \tag{13}$$

The $(0, \infty)$ interval transforms into the interval $(1, 0)$: again we shall denote the function

$$r\left(-\frac{1}{\sigma} \ln x\right)$$

by $r(x)$. Equation (1) takes the form

$$\sigma R(p) = \int_0^1 x^{(p/\sigma)-1} r(x)\, dx \tag{14}$$

from which we obtain with $p = (2k + 1)\sigma$,

$$\sigma R[(2k + 1)\sigma] = \int_0^1 x^{2k} r(x)\, dx. \tag{15}$$

Thus the value of the function $R(p)$ at the point $[(2k + 1)\sigma]$ gives the $2k$th moment of the function $r(x)$ in the $(0, 1)$ interval

It is known that the Legendre polynomials $P_k(x)$ form a complete orthogonal set in the $(-1, 1)$ interval; We extend the definition of $r(x)$ in the $(-1, 1)$ interval by making

$$r(-x) = r(x).$$

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This function, because of its evenness, can be expanded into a series of even Legendre polynomials. We thus have

$$r(x) = \sum_{k=0}^{\infty} C_k P_{2k}(x), \tag{16}$$

using the time scale we can write (16) in the form

$$r(t) = \sum_{k=0}^{\infty} C_k P_{2k}(e^{-\sigma t}). \tag{17}$$

To determine the coefficients C_k in (17) we observe that $P_{2k}(e^{-\sigma t})$, being an even polynomial in $e^{-\sigma t}$, of degree $2k$, will have as transform the function

$$\Phi_{2k}(p) = \frac{N(p)}{p(p + 2\sigma) \cdots (p + 2k\sigma)},$$

where $N(p)$ is a polynomial of degree less than $2k$. It is further known that

$$\int_0^1 x^{2n} P_{2k}(x)\, dx = 0 \quad \text{for} \quad n < k. \tag{18}$$

From Eqs. (18) and (15) follows that

$$\Phi_{2k}[(2n + 1)\sigma] = 0 \qquad n = 0, 1, \cdots, k - 1$$

hence the roots of $N(p)$ are

$$(2n + 1)\sigma \qquad n = 0, 1, \cdots, k - 1$$

and $\Phi_{2k}(p)$ can be written in the form

$$\Phi_{2k}(p) = \frac{(p - \sigma)(p - 3\sigma) \cdots [P - (2k - 1)\sigma]}{p(p + 2\sigma) \cdots (p + 2k\sigma)} A,$$

where A is a constant; to determine A we observe from the initial value theorem that

$$\lim_{p \to \infty} p\Phi_{2k}(p) = A = P_{2k}(1)$$

and since $P_{2k}(1) = 1$, we must have

$$A = 1.$$

Thus the Laplace transform of $P_{2k}(e^{-\sigma t})$ is given by

$$\Phi_{2k}(p) = \frac{(p - \sigma)(p - 3\sigma) \cdots [p - (2k - 1)\sigma]}{p(p + 2\sigma) \cdots (p + 2k\sigma)}. \tag{19}$$

Taking the transform of both sides of (17) we obtain

$$R(p) = \frac{C_0}{p} + \sum_{k=1}^{\infty} \frac{(p - \sigma) \cdots [p - (2k - 1)\sigma]}{p \cdots (p + 2k\sigma)} C_k . \tag{20}$$

If we replace p by

$$\sigma, 3\sigma, \cdots, (2k + 1)\sigma, \cdots$$

in Eq. (19), we obtain the system

$$
\begin{aligned}
\sigma R(\sigma) &= C_0 , \\
\sigma R(3\sigma) &= \frac{C_0}{3} + \frac{2C_1}{3\cdot 5} , \\
&\cdots\cdots\cdots\cdots \\
\sigma R[(2k+1)\sigma] &= \frac{C_0}{2k+1} + \frac{2kC_1}{(2k+1)(2k+3)} + \cdots \\
&\quad + \frac{2k(2k-2)\cdots 2C_k}{(2k+1)(2k+3)\cdots(4k+1)}.
\end{aligned} \tag{21}
$$

Again $R(\sigma)$ gives C_0, $R(3\sigma)C_1$ and so on. The partial sum $r_N(x)$ is the average of $r(x)$ with the Legendre kernel as the weighting factor. The constant σ is chosen with the same considerations as in *I*.

The above discussion furnishes a proof of the "Moment theorem" [1], [4]: that a function $r(x)$ in the (0, 1) interval is uniquely determined from its moments.

$$
M_m = \int_0^1 r(x)x^m \, dx \qquad m = 0, 1, \cdots .
$$

The proof is based on the orthogonality and completeness of the Legendre polynomials. In fact we also succeeded in writing $r(x)$ as an infinite sum of Legendre polynomials that can be determined from the moments of $r(x)$; these coefficients are given by the system (21) where on the left hand side we replace $R(2k+1)\sigma)$ by M_{2k}.

The Laguerre set. As a last case we shall consider the Laguerre set which has already been used in network analysis and synthesis [5]. The method described here will give a simpler way of determining the coefficients of the resulting expansion; it will also make clear the nature of the approximation, if the series contains only the first $N + 1$ terms.

The usual definition of the Laguerre polynomials $L_k(t)$ is

$$
L_k(t) = e^t \frac{d^k}{dt^k}\left[\frac{t^k}{k!} e^{-t}\right]. \tag{22}
$$

With

$$
\varphi_k(t) = e^{-t} L_k(t) \tag{23}
$$

we easily obtain for the transform of $\varphi_k(t)$

$$
\Phi_k(p) = \frac{p^k}{(p+1)^{k+1}}. \tag{24}
$$

Since the derivatives of $\Phi_k(p)$ of order less than k are zero at the origin, we must have [7]

$$
\int_0^\infty t^n \varphi_k(t) \, dt = 0 \quad \text{for} \quad n \leqq k - 1. \tag{25}
$$

With

$$
r(t) = \sum_{k=0}^{\infty} C_k \varphi_k(t) \tag{26}
$$

we have

$$R(p) = \sum_{k=0}^{\infty} C_k \frac{p^k}{(p+1)^{k+1}}. \tag{27}$$

It can be shown by differentiating n times the power series expansion at the origin of $1/(p + 1)$ that

$$\frac{p^k}{(p+1)^{k+1}} = p^k \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^n p^n. \tag{28}$$

Expanding the function $R(p)$ at the origin we obtain

$$R(p) = \sum_{k=0}^{\infty} a_k p^k. \tag{29}$$

From Eqs. (27), (28) and (29) we obtain equating equal powers of p

$$\begin{aligned} a_0 &= C_0 , \\ a_1 &= C_1 - C_0 , \\ &\cdots\cdots\cdots\cdots \\ a_k &= C_k - \binom{k}{1} C_{k-1} + \cdots + (-1)^k C_0 . \end{aligned} \tag{30}$$

The above system can be solved explicitly for C_k, with a simple induction [6]; the result is given by

$$C_k = \sum_{j=0}^{k} \binom{k}{j} a_{k-j} . \tag{31}$$

Thus knowing the coefficients a_k of the series expansion (29) of $R(p)$ we can readily determine from (31) the coefficients of (26).

Suppose that $r(t)$ is approximated by the finite sum

$$r_N(t) = \sum_{k=0}^{N} C_k \varphi_k(t) \tag{32}$$

of the first $N + 1$ terms of (26); then the transforms $R_N(p)$ and $R(p)$ of $r_N(t)$ and $r(t)$ have equal derivatives at the origin of order up to N, therefore [7]

$$\int_0^\infty t^n r_N(t)\, dt = \int_0^\infty t^n r(t)\, dt \qquad n \leq N \tag{33}$$

that is the function $r(t)$ and $r_N(t)$ have equal moments of order up to N.

Examples. In the following applications we shall use for our expansions the trigonometric set. We have approximated the inverse of $R(p)$ by

$$r_N(\theta) = \sum_{k=0}^{N} C_k \sin (2k + 1)\theta \tag{11}$$

where the coefficients C_k are given by (9) which we write in the form

$$\frac{\pi}{4} C_n = \sigma 2^{2n} R[(2n+1)\sigma] - \sum_{j=0}^{n-1} \left[\binom{2n}{j-1} - \binom{2n}{n-j-1} \right] C_j . \tag{34}$$

we have

$$R(p) = \sum_{k=0}^{\infty} C_k \frac{p^k}{(p+1)^{k+1}}. \tag{27}$$

It can be shown by differentiating n times the power series expansion at the origin of $1/(p+1)$ that

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$$\begin{aligned} a_0 &= C_0 , \\ a_1 &= C_1 - C_0 , \\ &\cdots\cdots\cdots\cdots \\ a_k &= C_k - \binom{k}{1} C_{k-1} + \cdots + (-1)^k C_0 . \end{aligned} \tag{30}$$

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$$\frac{\pi}{4} C_n = \sigma 2^{2n} R[(2n+1)\sigma] - \sum_{j=0}^{n-1} \left[\binom{2n}{j-1} - \binom{2n}{n-j-1} \right] C_j . \tag{34}$$

As examples we shall take functions whose inverse $r(t)$ is known, so as to compare $r(t)$ with $r_N(t)$. For the choice of σ we are guided either by the $(0, T)$ interval of interest, or from the $(0, p)$ interval of the real p axis in which $R(p)$ has its greatest variation; the choice of σ is not critical.

Example 1.

$$R(p) = \frac{\pi}{\pi} \frac{1}{(p + 0.2)^2 + 1} \quad \text{we take} \quad \sigma = 0.2$$

TABLE 2

	Example 1	Example 2
k	$C_k\, 10^4$	$C_k\, 10^4$
0	1724	1961
1	3154	4899
2	205	4009
3	−2075	460
4	380	633
5	530	1762
6	−754	166
7	474	862
8	−193	718
9	−40	199
10	58	982

From equation (34) we obtain for the coefficients C_k the numbers given in Table 2. These values inserted into (11) give for $r_N(\theta)$ at the points

$$\theta = 0, 5, \cdots, 90°$$

the numbers in Table 3.

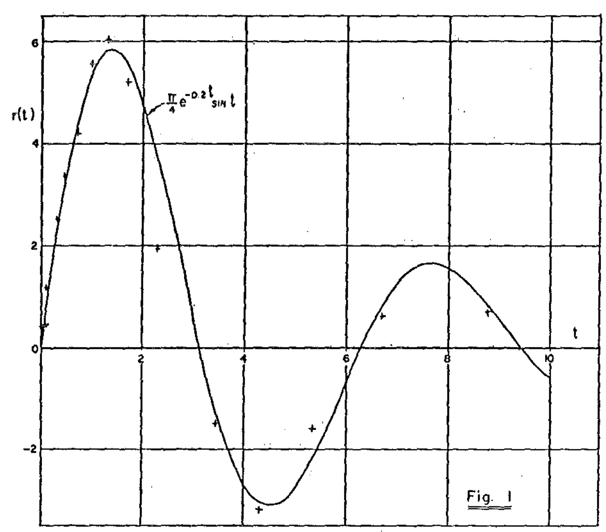

Fig. 1

The curve of Fig. 1 gives the inverse

$$r(t) = \frac{\pi}{4} e^{-0.2t} \sin t$$

of $R(p)$; the + points give the values of $r_N(t)$ as computed. The relationship between θ and t is established in (4).

Example 2.

$$R(p) = \frac{\pi}{4} \frac{1}{(p^2 + 1)^{1/2}} \qquad \sigma = 0.2$$

This example is chosen because of its discontinuity at the origin; clearly since

$$r(0) \neq 0$$

$r(\theta)$ will be discontinuous at $\theta = 0$, and $r_N(\theta)$ will exhibit the Gibb's phenomenon.

TABLE 3

	Example 1	Example 2
θ	$r_N(\theta) \times 10^4$	$r_N(\theta) \times 10^4$
5	398	8133
10	432	8739
15	1158	6958
20	2511	7787
25	3362	7896
30	4215	6363
35	5571	5977
40	6029	5241
45	5181	2612
50	4048	615
55	1944	−834
60	−1502	−3208
65	−3272	−3190
70	−1590	286
75	570	1748
80	694	−11
85	−33	−412

The values of C_k and $r_N(\theta)$ are listed in Table 2 and Table 3. In Fig. 2 the inverse

$$\frac{\pi}{4} J_0(t)$$

is plotted; the + points give the computed values of $r_N(t)$.

We see from the above examples that $r_N(t)$ is a good approximation of $r(t)$. The oscillation near $t = 0$ of Example 2 could have been avoided and a better fitting obtained if instead of $R(p)$ the function

$$R(p) - \frac{[pR(p)]_{p=\infty}}{p} = \frac{\pi}{4}\left(\frac{1}{(p^2 + 1)^{1/2}} - \frac{1}{p}\right)$$

were chosen, since its inverse satisfies the condition

$$r(0) = 0.$$

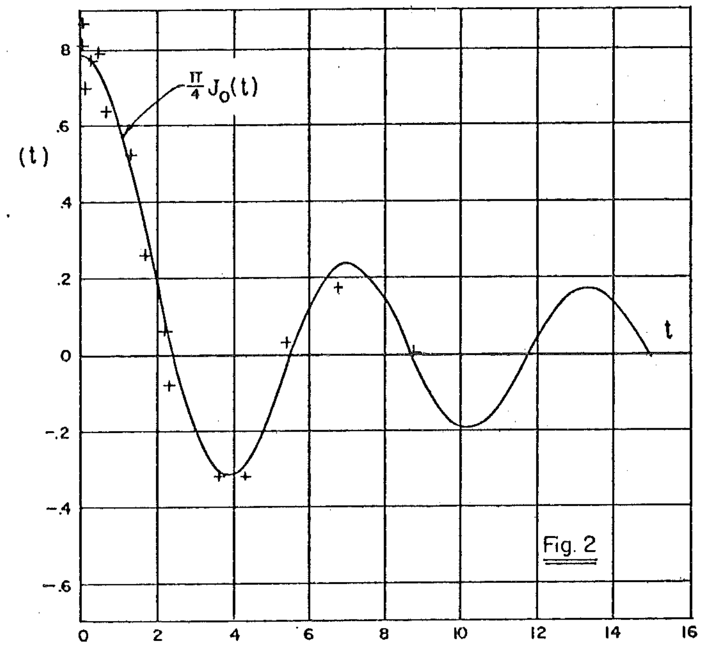

Fig. 2

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INVERSE
LAPLACE XFORM

TAUBARIAN THEOREM OF N. WIENER

K. YOSIDA, <u>FUNCTIONAL</u> <u>ANALYSIS</u>

(SPRINGER-VERLAG, BERLIN, 1965) p357

"LET $x(t) \in L^1(-\infty, \infty)$ BE SUCH THAT IT'S FOURIER TRANSFORM

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-i \cdot \xi \cdot t) dt$$

DOES NOT VANISH FOR ANY REAL ξ. THEN, FOR ANY $y(t) \in L_1(-\infty, \infty)$ AND $\varepsilon > 0$, WE CAN FIND REAL NUMBERS β's, THE COMPLEX NUMBER'S α's AND A POSITIVE INTEGER N IN SUCH A WAY THAT

$$\int_{-\infty}^{\infty} \left| y(t) - \sum_{j=1}^{N} \alpha_j \, x(t - \beta_j) \right| dt < \varepsilon$$

N. WIENER "TAUBARIAN THEOREMS"
ANN. OF MATH <u>33</u>, 1-100 (1932)

R.E.A.C. PALEY & N WIENER
"FOURIER TRANSFORMS IN THE COMPLEX DOMAIN", COLLOQ PUBL. AMER. MATH. SOC., 1934

N. WIENER, "THE FOURIER INTEGRAL & CERTAIN OF ITS APPLICATIONS" CAMBRIDGE, 1933.

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

29 March 1978

Prof. Gary L. Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Gary,

I have some new and interesting results in the signal-Hilbert transform sampling scheme that grew out of Szasz's theorem.

Our signal class consists of all real signals, $x(t)$, for which there exists an $a > 0$ such that

$$2X(f) \exp(af)\, \mu(f) \in L_2$$

where $\mu(\cdot)$ is the unit step function and $X(f)$ is the spectrum of $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)\, dt \,.$$

A subset of this class is all real L_2 bandlimited signals.

By Szasz's theorem, the basis set

$$e^{-af} \exp(-j2\pi ft_n)\, \mu(f) \tag{1}$$

is complete for $2X(f) \exp(af)\, \mu(f)$ iff

$$\sum_{n=1}^{\infty} \frac{a}{1 + |a + j2\pi t_n - \frac{1}{2}|^2} = \infty.$$

The sample times, $t_n = n$, are thus not applicable. Possible values of t_n include $n^{\frac{1}{2}}$ and $\exp(-n)$.

Let's suppose we have chosen a sample set $\{t_n\}$ and have applied a Gram-Schmidt orthonormalization to the Szasz basis elements in (1). Denote the mth orthonormal basis set by

$$A_m(f) = e^{-af} \sum_{n=1}^{m} c_{nm} \exp(-j2\pi ft_n)\, \mu(f) \,. \tag{2}$$

At worst, the coefficient c_{nm} could be numerically computed and then stored. We can now expand $2X(f)\, e^{af}\, \mu(f)$ in an orthonormal series:

$$2X(f)\, e^{af}\, \mu(f) = \sum_{m=1}^{\infty} \left(2X(f)\, e^{af}\, \mu(f) \,|\, A_m(f)\right) A_m(f) \qquad (3)$$

Using (2), the inner product can be written as

$$2 \sum_{n=1}^{m} c^*_{nm} \int_0^{\infty} X(f) \exp(j2\pi f t_n)\, df = \sum_{n=1}^{m} c^*_{nm}\, \hat{x}(t_n) \qquad (4)$$

where $\hat{x}(t)$ is the analytic signal corresponding to x:

$$\hat{x}(t) = x(t) + j\mathcal{H}[x(t)] \;.$$

Here, $\mathcal{H}(\cdot)$ denotes Hilbert transformation. Substituting (4) into (3) and using the identity

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} b_{nm} = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} b_{nm}$$

gives

$$X(f)\, \mu(f) = \frac{1}{2} e^{-af} \sum_{n=1}^{\infty} \hat{x}(t_n) \sum_{m=n}^{\infty} c^*_{nm}\, A_m(f) \;. \qquad (5)$$

From this, define the interpolation function

$$D_n(f) = \frac{1}{2} e^{-af} \sum_{m=n}^{\infty} c^*_{nm}\, A_m(f)\, \mu(f) \qquad (6)$$

so that (5) becomes

$$X(f)\, \mu(f) = \sum_{n=1}^{\infty} \hat{x}(t_n)\, D_n(f) \;. \qquad (7)$$

This is how we regain the half spectrum from the sampled analytic signal.

Let's investigate $D_n(f)$ further. Substituting (2) into (6) gives

$$D_n(f) = \frac{1}{2} e^{-2af} \sum_{m=n}^{\infty} c^*_{nm} \sum_{p=1}^{m} c_{pm} \exp(-j2\pi f t_p)\, \mu(f) \;.$$

Using the identity

$$\sum_{m=n}^{\infty} \sum_{p=1}^{m} b_{mp} = \sum_{p=1}^{\infty} \sum_{m=\max(p,n)}^{\infty} b_{mp}$$

gives

$$D_n(f) = \frac{1}{2} e^{-2af} \sum_{p=1}^{\infty} h_{np} \exp(-j2\pi f t_p)\, \mu(f)$$

where

$$h_{np} = \sum_{m=\max(p,n)}^{\infty} c^*_{nm}\, c_{pm} \tag{8}$$

Inverse transforming:

$$d_n(t) = \int_0^{\infty} D_n(f) \exp(j2\pi t f)\, df = \frac{j}{4\pi} \sum_{p=1}^{\infty} \frac{h_{np}}{t - (t_p - j\frac{a}{\pi})} . \tag{9}$$

(It might be possible to evaluate h_{np} directly from the t_n's without first computing the c_{nm}'s.) Interestingly, $d_n(t)$ is recognized as a countable number of poles on the complex t-plane at $\{t_p - j\frac{a}{\pi}\}$. The residue of the pth pole is $j(h_{np}/4\pi)$.

Using the Hermetian nature of X:

$$X(f) = X^*(-f) ,$$

we can write from (7)

$$X(f) = X(f)\mu(f) + X^*(-f)\mu(-f) = \sum_{n=1}^{\infty} \hat{x}(t_n) D_n(f) + \hat{x}^*(f_n) D_n^*(-f) .$$

Inverse transforming and recognizing that

$$\int_{-\infty}^{0} D_n^*(-f) \exp(j2\pi f t)\, df = d_n^*(t)$$

gives, after simplification,

$$x(t) = \sum_{n=1}^{\infty} x(t_n)\, \mathrm{Re}\, d_n(t) - H[x(t)]\Big|_{t=t_n} \mathrm{Im}\, d_n(t)$$

That's it! That's how we regain x(t) from the sampled analytic signal. It all rests on finding the c_{nm}'s in the Gram-Schmidt procedure (or the h_{np}'s) for a given sample time set $\{t_n\}$.

One final challenge (in the real world) is sampling the signal's Hilbert transform. Seems as if this might be done to a "good" approximation by sampling the input times t_n and approximating the Hilbert transform by a coordinate distorted version of Sabri and Steenhaart's Hilbert transform matrix.*

Let me know what you think.

Best personal regards,

Bob

Robert J. Marks II
Assistant Professor

RM:bb

*M. S. Sabri and W. Steenhaart, "Discrete Hilbert Transform Filtering," IEEE Transactions on Acousitcs, Speech and Signal Processing, ASSP-25, p. 452, 1977.

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

18 May 1978

Professor Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Gary,

A quick note to update some filter-and-sample scheme results.

Consider the filter in figure 1 (from Papoulis). f_0 is a given frequency constant. One can easily show that

$$y(t) = \int_0^t u(\tau) \exp(-j2\pi f_0^2 t\tau)\, d\tau \, .$$

The system is then kind of a Fourier transformer.

Let $u(t) \in L_2[0,b]$. Then, for $t \geq b$, we get

$$y(t) = \int_0^b u(\tau) \exp(-j2\pi f_0^2 t\tau)\, d\tau \quad ; \; t \geq b \, .$$

Thus, if we define the Fourier transform

$$U(f) = \int_{-\infty}^{\infty} u(t) \exp(-j2\pi ft)\, dt \, ,$$

then

$$y(t) = U(f_0^2 t) \quad ; \quad t \geq b$$

and

$$U(f) = y(f/f_0^2) \; ; \quad f \geq bf_0^2 \, .$$

By Shannon's sampling theorem, the sample values $\{U(n/b) | \; n=0,\pm1,\pm2,...\}$ completely specify $u(t)$. If we further restrict $u(t)$ to be real, then $U(f) = U^*(-f)$ and $\{U(n/b) \mid n=0,1,2,...\}$ completely specifies $u(t)$. We can get these samples by sampling $y(t)$:

$$U(\tfrac{n}{b}) = y(\tfrac{n}{bf_o^2}) \quad ; \quad n \geq b^2 f_o^2 \ .$$

We can get the $n = 1,2,3...$ samples by simply requiring that $f_o^2 b^2 \leq 1$. The $n = 0$ sample can be obtained elsewhere by an integration.

This filter, then, is kind of an analog FFT processor for short pulses. Kind of neat.

In practice, implementing the complex valued filter and chirp modulators in figure 1 might be a problem. An alternate filter is shown in figure 2. Here the input-output relation is kind of a Laplace transform:

$$y(t) = \int_0^t u(\tau)\, e^{-s_o^2 t\tau}\, d\tau \ ,$$

where s_o^2 is constant. Again, let $u(t) \in L_2[0,b]$. Then for $t \geq b$,

$$y(t) = \hat{U}(s_o^2/t) \quad ; \quad t \geq b \ ,$$

where $\hat{U}$ is the Laplace x form of u:

$$\hat{U}(s) = \int_0^\infty u(t)\, e^{-st}\, dt \ .$$

From Szasz's theorem, we can sample $\hat{U}(s)$ in a number of ways that will uniquely characterize $u(t)$. If desired, we could obtain Fourier coefficients or Legendre coefficients from the vector of Laplace transform samples by a simple matrix transformation.

Best wishes,

Bob

Robert J. Marks II
Assistant Professor

RM:bb

P.S. Doug tells me he's been snowed in his job lately but will send me a draft of the Laplace II paper by next week.

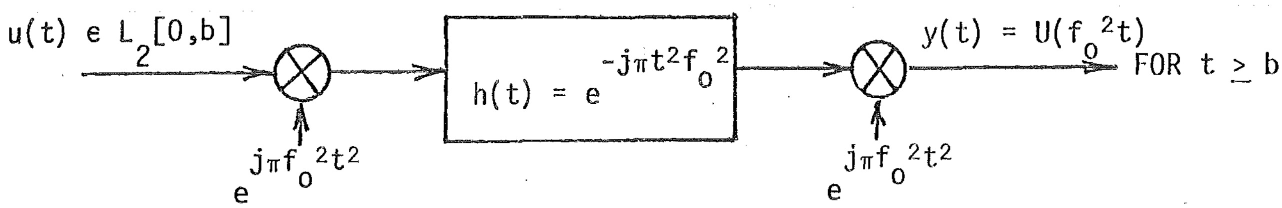

FIGURE 1

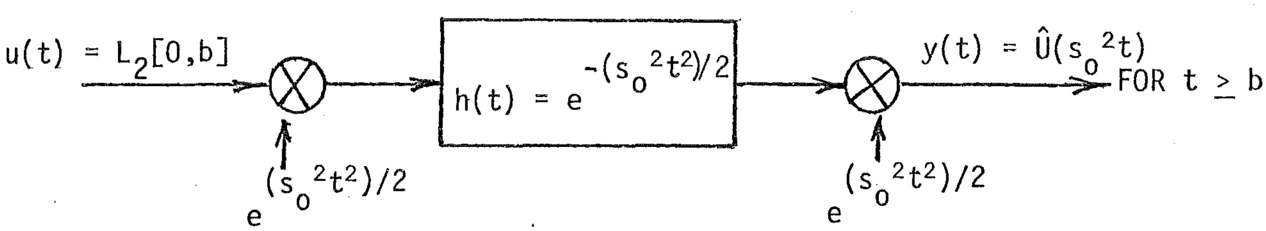

FIGURE 2

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

Department of Electrical Engineering

16 June 1978

Dr. Gary Wise
Department of Electrical Engineering
University of Texas
Austin, Texas 78712

Dear Gary,

The following is a derivation of the procedure for regaining an L_2^+ signal from its uniformly sampled Laplace transform. Some theorems are also given.

We begin with the orthogonality of the Legendre polynomials:

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\,\delta_{nm} .$$

Making the variable substitution

$$x = 2e^{-rt} - 1 \quad ; \; r > 0$$

gives the following orthonormal basis set elements:

$$[r(2m+1)]^{\frac{1}{2}}\, e^{-\frac{rt}{2}}\, P_m[2e^{-rt} - 1] \quad ; \; m = 0,1,2,\ldots$$

Szasz's theorem, however, requires indexing to begin at unity. Thus, setting $n = m+1$, we obtain

$$\phi_n(t) = [r(2n-1)]^{\frac{1}{2}}\, e^{-\frac{rt}{2}}\, P_{n-1}[2e^{-rt} - 1] \quad ; \; n = 1,2,3,\ldots$$

From Szasz's theorem, $\{\phi_n(t) \mid n = 1,2,3,\ldots\}$ is complete on L_2^+ since, for every $n \geq 1$, there exists a unique set $\{b_{qn} \mid q = 1,2,\ldots,n\}$ such that

$$\exp[-(n - \tfrac{1}{2})rt] = \sum_{q=1}^{n} b_{qn}\, \phi_q(t) .$$

The basis elements, $\exp[-a_n t]$, are complete in L_2^+ since

$$a_n = (n - \tfrac{1}{2})r > 0 \quad \text{for all positive } n$$

and

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re} a_n}{1 + |a_n - \tfrac{1}{2}|^2} = \infty .$$

Let $x(t) \in L_2^+$. Then

$$x(t) = \sum_{n=1}^{\infty} \alpha_n \, \phi_n(t) ,$$

where

$$\alpha_n = \int_0^{\infty} x(t) \, \phi_n(t) \, dt .$$

Using the expression

$$P_{n-1}(t) = \frac{1}{2^{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^k (2n-2k-2)!}{k!(n-k-1)!(n-2k-1)!} \, t^{n-2k-1}$$

gives

$$\alpha_n = \frac{[r(2n-1)]^{\frac{1}{2}}}{(+2)^{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^k (2n-2k-2)!}{k!(n-k-1)!(n-2k-1)!}$$

$$\cdot \int_0^{\infty} e^{-\frac{rt}{2}} [2e^{-rt} - 1]^{n-2k-1} \; x(t) \, dt .$$

Exapanding the integrand into a binomial series gives

$$\alpha_n = \frac{[r(2n-1)]^{\frac{1}{2}}}{(-2)^{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^k (2n-2k-2)!}{k!(n-k-1)!}$$

$$\cdot \sum_{q=0}^{n-2k-1} \frac{(-2)^q}{q!(n-2k-q-1)!} \; X[r(q+\tfrac{1}{2})] \; ,$$

where the Laplace transform of the signal is defined by:

$$X(s) = \int_0^\infty x(t) \, e^{-st} \, dt \, .$$

Using the identity

$$\sum_{k=0}^{[\frac{n-1}{2}]} \; \sum_{q=0}^{n-2k-1} \; = \; \sum_{q=0}^{n-1} \; \sum_{k=0}^{[\frac{n-q-1}{2}]}$$

gives

$$\alpha_n = \frac{[r(2n-1)]^{\frac{1}{2}}}{(-2)^{n-1}} \sum_{q=0}^{n-1} \frac{(-2)^q}{q!} \; X[r(q+\tfrac{1}{2})]$$

$$\cdot \sum_{k=0}^{[\frac{n-q-1}{2}]} \frac{(-1)^k (2n-2k-1)!}{k!(n-k-1)!(n-2k-q-1)!} \, .$$

Substituting into the orthonormal expansion and using the identity

$$\sum_{n=1}^{\infty} \; \sum_{q=0}^{n-1} \; = \; \sum_{q=0}^{\infty} \; \sum_{n=q+1}^{\infty}$$

gives the final result:

$$x(t) = r \sum_{q=0}^{\infty} X[r(q+\tfrac{1}{2})] \; I_q(rt) \; ,$$

$I_q(t) = \frac{(-1)^q}{(q!)^2} e^{-t/2} \sum_{n=q}^{\infty} (-1)^n (2n+1) \frac{(n+q)!}{(n-q)!} P_n[2e^{-t} - 1]$

where our interpolation function is

$$I_q(t) = e^{-\frac{t}{2}} \frac{(-2)^q}{q!} \sum_{n=q+1}^{\infty} \frac{(2n-1)}{(-2)^{n-1}} P_{n-1}[2e^{-t} - 1]$$

$$\cdot \sum_{k=0}^{[\frac{n-q-1}{2}]} \frac{(-1)^k (2n-2k-2)!}{k!(n-k-1)!(n-2k-q-1)!} .$$

This is the simplest version I've found though I'm sure with some "lengthy but straightforward manipulations," it can be placed in a better form.

Following are some theorems:

THEOREM 1: A Generalized Method of Expansion.

Let τ be a given constant. Then

$$x(t) = r \sum_{q=0}^{\infty} X[r(q+\tfrac{1}{2})] \exp[-r(q+\tfrac{1}{2})\tau] \, I_q[r(t+\tau)] .$$

[NOTE: Our original interpolation of the signal was for $\tau = 0$.]

<u>Proof</u>:

From the shift property of the Laplace transform:

$$\mathcal{L}[x(t-\tau)] = X(s) e^{-s\tau} \quad ; \; x(t) \in L_2^+$$

Thus:

$$x(t-\tau) = r \sum_{q=0}^{\infty} X[r(q+\tfrac{1}{2})] \exp[-r(q+\tfrac{1}{2})\tau] \, I_q[rt] .$$

Letting $t = t+\tau$ completes the proof.

<u>Lemma</u>: The inner product of two basis elements, $rI_p(rt)$ and $rI_q(rt)$, is given by the expression:

$$r^2 \int_0^{\infty} I_p(rt) \, I_q(rt) \, dt = r \, \beta_{qp} ,$$

where

$$\beta_{qp} = \frac{(-2)^{p+q}}{q!p!} \sum_{n=\max(p,q)+1}^{\infty} c_{nq}\, c_{np} \frac{(2n-1)}{(-2)^{2n-2}}$$

and

$$c_{nq} = \sum_{k=o}^{[\frac{n-q-1}{2}]} \frac{(-1)^k (2n-2k-2)!}{k!(n-k-1)!(n-2k-q-1)!} \,.$$

The proof follows directly from the orthogonality of the Lengendre polynomials. Note that

$$\beta_{pq} = \beta_{qp} \,.$$

Note also that β_{qp} is independent of our sampling rate index, r.

THEOREM (Parseval):

The squared L_2 norm (energy) of x(t) can be written as

$$E = \int_0^{\infty} |x(t)|^2\, dt = r \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \bar{X}_q\, \beta_{qp}\, X_p \,,$$

where

$$X_p \triangleq X[r(p+\tfrac{1}{2})]$$

and the overbar denotes complex conjugate. In matrix form:

$$E = r\, \bar{X}^T B\, X \,,$$

where X denotes the (infinite) column vector of sample Laplace values and B is the square matrix that contains the offline computed β_{pq}'s. This relation will help in determining the required number of Laplace samples.

<u>Corollary</u>: Sample Energy Updating.

Let

$$E_N = r \sum_{p=1}^{N} \sum_{q=1}^{N} \bar{X}_q\, \beta_{qp}\, X_p$$

denote the "energy" associated with N Laplace samples. Obviously, from the previous theorem,

$$E = \lim_{N\to\infty} E_N = \int_0^\infty |x(t)|^2 \, dt \, .$$

E_N can be updated by the following relation:

$$E_{N+1} = E_N + 2r \sum_{q=1}^{N} \beta_{q,N+1} \, \mathrm{Re}\,[\bar{X}_q X_{N+1}] + r \, |X_{N+1}|^2 \, \beta_{N+1,N+1} \, ,$$

where Re(·) denotes "the real component of." Note that

$$E_1 = |X_1|^2 \, \beta_{1,1} \, .$$

Proof:

$$E_{N+1} = r \sum_{p=1}^{N+1} \sum_{q=1}^{N+1} \bar{X}_q \, \beta_{qp} \, X_p$$

$$= E_N + r \, X_{N+1} \sum_{q=1}^{N+1} \bar{X}_q \, \beta_{q,N+1} + r \, \bar{X}_{N+1} \sum_{p=1}^{N} X_p \, \beta_{N+1,p} \, .$$

Taking advantage of the fact that $\beta_{pq} = \beta_{qp}$, we combine the summations and arrive at the desired results. Updating E_N will tell us how many more samples we need.

That's all for now. The fascinating part to me is the arbitrary nature of the sample rate parameter, r. Let me know what you think.

Best wishes,

Bob

Robert J. Marks II
Assistant Professor

RM:bb